$$\frac{1}{N}\left\{ (xD)^2 - 2u(xD) + u^2 \right\} F(x) \Big|_{x=1}$$

$$= \frac{1}{N}\left\{ xD\big(xF'(x)\big) - 2ux\,F'(x) + u^2 F(x) \right\}\Big|_{x=1}$$

$$= \frac{1}{N}\left\{ x\left[xF'' + F'(x)\right] - 2ux\,F'(x) + u^2 F(x) \right\}\Big|_{x=1}$$

$$= \frac{1}{N}\left\{ x^2 F'' + xF'(x) - 2ux\,F'(x) + u^2 F(x) \right\}\Big|_{x=1}$$

$$= \frac{1}{N}\left\{ x^2 F'' + x(1-2u)F' + u^2 F \right\}\Big|_{x=1}$$

$$= \frac{F''(1) + (1-2u)F'(1) + u^2 F(1)}{F(1)}$$

$$= \frac{F''(1)}{F(1)} + (1-2u)\frac{F'(1)}{F(1)} + u^2$$

$$= \frac{F''(1)}{F(1)} + (1-2u)u + u^2 = \frac{F''(1)}{F(1)} + u - u^2$$

$$= \frac{F''(1)}{F(1)} + \frac{F'(1)}{F(1)} - \left(\frac{F'(1)}{F(1)}\right)^2$$

$$(\log F)' = \frac{F'(x)}{F(x)}$$

$$(\log F)'' = \left(\frac{F'(x)}{F(x)}\right)' = \frac{F''}{F} + \frac{F' \; F'}{(-1)F^2} = \frac{F''}{F} - \left(\frac{F'}{F}\right)^2$$

$\therefore$ Above becomes

$$= \frac{F''(1)}{F(1)} - \left(\frac{F'(1)}{F(1)}\right)^2 + \frac{F'(1)}{F(1)}$$

$$= \left(\ln F(x)\right)''\Big|_{x=1} + \left(\ln F(x)\right)'\Big|_{x=1}$$

$$= \left\{ \left(\ln F(x)\right)' + \left(\ln F(x)\right)'' \right\}\Big|_{x=1}$$ eq 4.1.3 ✓

$$\sum_{n,k} h(n,k)\frac{x^n}{n!}y^k = e^{yD(x)}$$

$h(n)$ = # of hands of weight n.

$h(n,k)$ = # of hands of weight n made up of k cards

$$\sum_{n,k} h(n,k)\frac{x^n}{n!}y^k = e^{yD(x)}$$

$\left.\frac{\partial}{\partial y}\right|_{y=1}$ →

(1) $$\sum_{n,k} h(n,k)\frac{x^n}{n!}k\,y^{k-1} = D(x)e^{yD(x)}$$

(2) $$\sum_{n,k} h(n,k)\frac{x^n}{n!}k = D(x)\left.e^{yD(x)}\right|_{y=1} = D(x)H(x)$$

→ $$\sum_{n}\frac{x^n}{n!}\sum_{k} h(n,k)\,k$$

$$\mu(n) = \left[\frac{h(n)x^n}{n!}\right]D(x)H(x)$$

$$\mu(n) = \left[\;\right]\left(\sum_{r\geq 0}\frac{d_r}{r!}x^r\right)\left(\sum_{n\geq 0}\frac{h(n)}{n!}x^n\right)$$

$$= \sum_{r,n\geq 0}\frac{d_r h_n}{r!\,n!}x^{r+n}$$

$$u(n) = \left[\frac{h(n)x^n}{n!}\right] \sum_{r,n\geq 0} \frac{d_r h_n}{r!\,n!} x^{r+n}$$

$p = r+n$

$r = p-n$; $n = p-r$

$$= \left[\frac{h(n)\,x^n}{n!}\right] \sum_{p\geq 0} \sum_{n\geq 0} \frac{d_{p-n}\,h_n}{(p-n)!\,n!} x^p$$

$$= \left[\frac{h(n)\,x^n}{n!}\right] \sum_{p\geq 0} \sum_{r\geq 0} \frac{d_r\,h_{p-r}}{r!\,(p-r)!} x^p$$

$$= \left[\frac{h(n)\,x^n}{n!}\right] \sum_{n\geq 0} \sum_{r\geq 0} \frac{d_r\,h_{n-r}}{r!\,(n-r)!} x^n$$

$$= \frac{n!}{h(n)} \sum_{r\geq 0} \frac{d_r\,h_{n-r}}{r!\,(n-r)!} = \frac{1}{h(n)} \sum_{r\geq 0} \binom{n}{r} d_r\, h(n-r) \qquad \text{eq 4.1.6}$$

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$$u(n) = \frac{1}{n!}\sum_{r}\binom{n}{r}(r-1)!(n-r)!$$

$$= \sum_{r=} \frac{\cancel{n!}\,(r-1)!\,\cancel{(n-r)}!}{\cancel{n!}\,\cancel{(n-r)}!\,r!} = \sum_{r=1}^{n}\frac{1}{r} = H_n$$

$F(x) =$

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$$\mu(n) = \frac{1}{n!}\sum_r \binom{n}{r}(r-1)!\,(n-r)!$$

$$F(x) = x(x+1)(x+2)\cdots(x+n-1)$$

$$F(1) = 1(2)(3)\cdots(n) = n!$$

$$\log F(x) = \sum_{k=0}^{n-1}\log(x+k)$$

$$\frac{d}{dx}(\log F(x)) = \sum_{k=0}^{n-1}\frac{1}{x+k} = ?$$

$$\frac{d}{dx}(\log F(x))\Big|_{x=1} = \sum_{k=0}^{n-1}\frac{1}{1+k} = \sum_{k=1}^{n}\frac{1}{k} \equiv H_n$$

$$\frac{d^2}{dx^2}(\log F(x)) = -\sum_{k=0}^{n-1}\frac{1}{(x+k)^2} \Rightarrow \frac{d^2}{dx^2}(\log F(x))\Big|_{x=1} = -\sum_{k=0}^{n-1}\frac{1}{(1+k)^2}$$

$$= -\sum_{k=1}^{n}\frac{1}{k^2}$$

$$\sigma^2 = \left\{(\log F)' + (\log F)''\right\}_{x=1} = \left\{H_n - \sum_{k=1}^{n}\frac{1}{k^2}\right\}$$

$$= \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k^2}\right) =$$

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$$\sum_{r\geq 0}\binom{r}{k}x^r = ? \equiv S_k(x) \quad \text{what is } S_k(x) = ?$$

$$y^k\sum_{r\geq 0}\binom{r}{k}x^r = \sum_k y^k S_k(x)$$

sum over all $-\infty$ to ∞ of the index k

$$\sum_k y^k S_k(x) = \sum_k y^k \sum_{r\geq 0}\binom{r}{k}x^r$$

$$= \sum_{r\geq 0} x^r \underbrace{\sum_k \binom{r}{k} y^k}_{(1+y)^r}$$

$$= \sum_{r\geq 0} x^r(1+y)^r = \frac{1}{1-(x(1+y))}$$

$$= \frac{1}{1-x-xy} =$$

$$= \frac{1}{1-x}\,\frac{1}{\left(1-\frac{x}{1-x}\cdot y\right)} = \frac{1}{1-x}\sum_{k\geq 0}\left(\frac{x}{1-x}\right)^k y^k$$

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$$\therefore \quad S_k(x) = \frac{x^k}{(1-x)^{k+1}} \quad \text{eq } 4.3.1$$

$$= \sum_{k\geq 0} x^k \sum_{n} \binom{k}{n-k} x^{n-k}$$

$$= \sum_{k\geq 0} x^k \underbrace{\sum_{r} \binom{k}{r} x^r}_{(1+x)^k}$$

$$= \sum_{k\geq 0} x^k (1+x)^k = \frac{1}{1-x(1+x)}$$

$$\sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = f(n)$$

$$F(x) = \sum_{n\geq 0} x^n \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$$

$$= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} \sum_{n\geq 0}\binom{n+k}{m+2k} x^n$$

$$= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \underbrace{\sum_{n\geq 0}\binom{n+k}{m+2k} x^{n+k}}$$

$$\frac{x^{m+2k}}{(1-x)^{m+2k+1}} \quad \text{By eq 4.3.1}$$

$$= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1}\frac{x^{m+k}}{(1-x)^{m+2k+1}}$$

$$= \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1}\frac{x^k}{(1-x)^{2k}}$$

$$= \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{2k}{k}\left[\frac{(-x)^k}{(1-x)^2}\right]^k \frac{1}{k+1}$$

$$= \frac{x^m}{(1-x)^{m+1}} \frac{\left(1-\sqrt{1-4\left(\frac{-x}{(1-x)^2}\right)}\right)}{2\left(\frac{-x}{(1-x)^2}\right)}$$

$$= \frac{-x^{m-1}}{2(1-x)^{m-1}}\left[1-\sqrt{1+\frac{4x}{(1-x)^2}}\right]$$

$$\frac{(1-x)^2+4x}{(1-x)^2} = \frac{1-2x+x^2+4x}{(1-x)^2}$$

$$= \cancel{\boxed{1-x}}^{\,2} \quad \frac{1+2x+x^2}{(1-x)^2}$$

$$= \frac{(1+x)^2}{(1-x)^2}$$

$\therefore$

$$= \frac{-x^{m-1}}{2(1-x)^{m-1}}\left[1-\frac{1+x}{1-x}\right]$$

/* Assuming both terms are positive */

$$= \frac{-x^{m-1}}{2(1-x)^{m-1}}\left[\frac{\cancel{1}-x-(\cancel{1}+x)}{1-x}\right] = \frac{\cancel{-}x^{m-1}}{\cancel{2}(1-x)^{m-1}}\frac{(\cancel{-2}x)}{(1-x)}$$

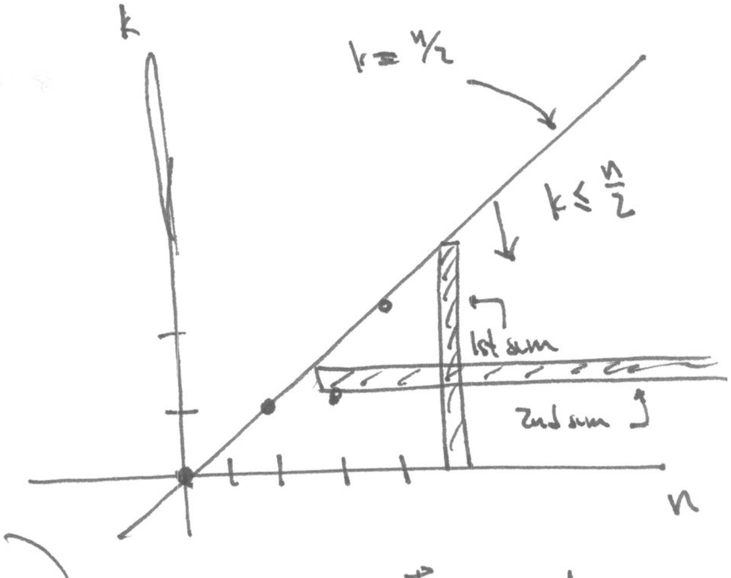
$$= \frac{x^m}{(1-x)^m} = x\frac{x^{m-1}}{(1-x)^m} = x\sum_{r\geq 0}\binom{r}{m-1}x^r$$

$$= \sum_{r\geq 0}\binom{r}{m-1}x^{r+1}$$

so $[x^n]\left(\sum_{r\geq 0}\binom{r}{m-1}x^{r+1}\right)$

$= \binom{n-1}{m-1}$

$$f_n = \sum_{k \le n/2} (-1)^k \binom{n-k}{k} y^{n-2k}$$



Changing order of summation.

$$F(x) = \sum_{n \ge 0} x^n \sum_{k \le n/2} (-1)^k \binom{n-k}{k} y^{n-2k}$$

$$= \sum_{n \ge 0} \sum_{k \le n/2} (-1)^k \binom{n-k}{k} x^n y^n y^{-2k}$$

$$= \sum_{k \ge 0} \sum_{n \ge 2k} (-1)^k \binom{n-k}{k} x^n y^n y^{-2k}$$

$$= \sum_{k \ge 0} (-1)^k y^{-2k} \underbrace{\sum_{n \ge 2k} \binom{n-k}{k} x^n y^n}$$

$$\sum_{n \ge 2k} \binom{n-k}{k} (xy)^{n-k+k}$$

$$= \sum_{k \ge 0} (-1)^k y^{-2k} (xy)^k \underbrace{\sum_{n \ge 2k} \binom{n-k}{k} (xy)^{n-k}}$$

For $n < 2k$ the binomial coefficient is zero.

$$= \qquad \frac{(xy)^k}{(1-xy)^{k+1}}$$

$$= \sum_{k \geq 0} (-1)^k \cancel{y^{-k}} x^k \frac{x^k \cancel{y^k}}{(1-xy)^{k+1}}$$

$$= \sum_{k \geq 0} (-1)^k \frac{x^{2k}}{(1-xy)^{k+1}} = \frac{1}{(1-xy)} \sum_{k \geq 0} \left[\frac{-x^2}{(1-xy)}\right]^k$$

$$= \frac{1}{1-xy} \frac{1}{\left(1-\left(\frac{-x^2}{1-xy}\right)\right)} = \frac{1}{1-xy+x^2}$$

$$y \to x + \frac{1}{x} \qquad y = \frac{x^2+1}{x}$$

$$\sqrt{y^2-4} \longrightarrow \sqrt{x^2+2+\frac{1}{x^2}-4} = \sqrt{x^2-2+\frac{1}{x^2}}$$

$$= x - \frac{1}{x}$$

$$\sum_{k \leq \frac{n}{2}} (-1)^k \binom{n-k}{k} y^{n-2k} = \sum_{k \leq n/2} (-1)^k \binom{n-k}{k} (x^2+1)^{n-2k} x^{-(n-2k)}$$

$$= x^{-n} \sum_{k \leq n/2} (-1)^k \binom{n-k}{k} (x^2+1)^{n-2k} x^{2k}$$

$$= \frac{1}{\left(x-\frac{1}{x}\right)} \left\{ \left(\frac{x+\frac{1}{x}+x-\frac{1}{x}}{2}\right)^{n+1} - \left(\frac{x+\frac{1}{x}-x+\frac{1}{x}}{2}\right)^{n+1} \right\}$$

$$= \frac{x}{x^2-1}\left\{ x^{n+1} - \frac{1}{x^{n+1}} \right\}$$

$$= \frac{x^{2n+2} - 1}{x^n(x^2-1)}$$

So $\Rightarrow$

$$\sum_{k \le n/2} (-1)^k \binom{n-k}{k} (1+x^2)^{n-2k} x^{2k} = \frac{x^{2n+2}-1}{x^2-1}$$

$t = x^2$

$$\sum_{k \le n/2} (-1)^k \binom{n-k}{k} (1+t)^{n-2k} t^k = \frac{t^{n+1}-1}{t-1} \qquad \text{eq } 4.3.7$$

$t = 1$

$$\Rightarrow \sum_{k \le n/2} (-1)^k \binom{n-k}{k} 2^{n-2k} = (n+1)\left(\frac{t^n}{1}\right)\Bigg|_{t=1} = n+1$$

eq 4.3.7

$$\sum_{k \le n/2} (-1)^k \binom{n-k}{k} \left(\sum_{p \ge 0} \binom{n-2k}{p} t^p \right) t^k = \sum_{p=0}^{n} t^p$$

$$= \sum_{p \ge 0} \sum_{k \le n/2} (-1)^k \binom{n-k}{k} \binom{n-2k}{p} t^{p+k} = \sum_{p=0}^{n} t^p$$

let $m \equiv p+k$ Then $p = m-k$

$$\sum_{m \ge k} \left(\sum_{k \le n/2} (-1)^k \binom{n-k}{k} \binom{n-2k}{m-k} \right) t^m = \sum_{m=0}^{n} t^m$$

$$\Rightarrow \sum_{k \le n/2} (-1)^k \binom{n-k}{k} \binom{n-2k}{m-k} = \begin{cases} 1 & \text{if } 0 \le m \le n \\ 0 & \text{otherwise} \end{cases}$$

let $n=2$

$$\sum_{k \le 1} (-1)^k \binom{2-k}{k} \binom{2-2k}{m-k} = \begin{cases} 1 & 0 \le m \le 2 \\ 0 & \text{other} \end{cases}$$

$$\overset{k=0}{1 \cdot 1} \binom{0}{m} + (-1) \cdot 1 \binom{0}{m-1} = \binom{0}{m} - \binom{0}{m-1} = \begin{cases} 1 & m=0, \\ -1 & m=1 \\ 0 & \text{other.} \end{cases} \; ?$$

$$f_n = \sum_k \binom{n+k}{2k} 2^{n-k} \qquad n \ge 0$$

$$F(x) = \sum_{n\ge0} x^n \sum_k \binom{n+k}{2k} 2^{n-k} = \sum_k 2^{-k} \sum_{n\ge0}\binom{n+k}{2k}(2x)^n$$

$$= \sum_k 2^{-k}\cdot \cancel{x}(2x)^{-k}\sum_{n\ge0}\binom{n+k}{2k}(2x)^{n+k}$$

Now $\sum_{n\ge0}\binom{n+k}{2k}(2x)^{n+k} = \frac{(2x)^{2k}}{(1-2x)^{2k+1}}$ so clow is

$$= \cancel{\sum_k 2^{-k}}(2 \qquad \sum_k (4x)^{-k}\frac{(2x)^{2k}}{(1-2x)^{2k+1}}$$

$$= \frac{1}{(1-2x)}\sum_k \frac{\cancel{4^k}\cdot\cancel{4^{-k}}\, x^{-k}\cdot x^{2k}}{(1-2x)^{2k}} = \frac{1}{(1-2x)}\sum_k \frac{x^k}{((1-2x)^2)^k}$$

$$= \frac{1}{(1-2x)}\sum_k \left[\frac{x}{(1-2x)^2}\right]^k = \frac{1}{(1-2x)}\,\frac{1}{\left[1-\frac{x}{(1-2x)^2}\right]}$$

$$= \frac{1}{1-2x}\,\frac{(1-2x)^2}{((1-2x)^2-x)} = \frac{1-2x}{1-4x+4x^2-x}$$

$$= \frac{1-2x}{1-5x+4x^2} = \frac{1-2x}{(1-4x)(1-1x)}$$

$$= \frac{A}{1-4x} + \frac{B}{1-x}$$

$$A = \left.\frac{1-2x}{1-x}\right|_{x=1/4} \qquad B = \left.\frac{1-2x}{1-4x}\right|_{x=1}$$

$$= \frac{1-1/2}{1-1/4} \qquad B = \frac{-1}{-3} = \frac{1}{3}$$

$$= \frac{1/2}{3/4} = \frac{2}{3}$$

$\therefore$

$$F(x) = \frac{2}{3(1-4x)} + \frac{1}{3(1-x)}$$

$$= \frac{2}{3}\sum_{k\geq 0}(4x)^k + \frac{1}{3}\sum_{k\geq 0} x^k = \sum_{k\geq 0}\left(\frac{2}{3}(4^k) + \frac{1}{3}\right)x^k$$

$$\therefore \sum_k \binom{n+k}{2k} 2^{n-k} = \frac{2^{2n+1}}{3} + \frac{1}{3}$$

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$$f_n(y) = \sum_k \binom{n}{k}\binom{2k}{k} y^k \qquad n \geq 0$$

$$F(x,y) = \sum_{n\geq 0} f_n x^n$$

$$= \sum_{n\geq 0} x^n \sum_k \binom{n}{k}\binom{2k}{k} y^k = \sum_k \binom{2k}{k} y^k \underbrace{\sum_{n\geq 0}\binom{n}{k} x^n}_{\frac{x^k}{(1-x)^{k+1}}}$$

$$= \frac{1}{(1-x)} \sum_k \binom{2k}{k} \frac{(xy)^k}{(1-x)^k}$$

$$= \frac{1}{(1-x)} \sum_k \binom{2k}{k} \left(\frac{xy}{1-x}\right)^k = \frac{1}{(1-x)} \frac{1}{\sqrt{1-4\left(\frac{xy}{1-x}\right)}}$$

$$= \frac{1}{(1-x)} \frac{\sqrt{1-x}}{\sqrt{1-x-4xy}}$$

$$= \frac{1}{\sqrt{(1-x)(1-x-4xy)}} = \frac{1}{\sqrt{(1-x)(1-x(1+4y))}}$$

$y = -1/4$ $\quad F(x) = \dfrac{1}{\sqrt{1-x}}$

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = \sum_{k\geq 0} \binom{-1/2}{k} (-1)^k x^k$$

$$= \cdots$$

$y = -1/2$ then

$$F(x) = \frac{1}{\sqrt{(1-x)(1-x(1+-2))}} = \frac{1}{\sqrt{(1-x)(1+x)}} = \frac{1}{\sqrt{1-x^2}}$$

$$= (1-x^2)^{-1/2} = \sum_{k\geq 0} \binom{-1/2}{k} (-1)^k x^{2k}$$

Pr:

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k}2^k \qquad m,n \geq 0$$

$$\sum_{n\geq 0} x^n \sum_k \binom{m}{k}\binom{n+k}{m} = \sum_{n\geq 0} x^n \sum_k \binom{m}{k}\binom{n}{k} 2^k \qquad m,n \geq 0$$

$$\Rightarrow \sum_k \binom{m}{k} \sum_{n\geq 0} \binom{n+k}{m} x^n = \sum_k \binom{m}{k} 2^k \underbrace{\sum_{n\geq 0} x^n \binom{n}{k}}_{\frac{x^k}{(1-x)^{k+1}}}$$

$$\Rightarrow \sum_k \binom{m}{k} x^{-k} \sum_{n\geq 0} \binom{n+k}{m} x^{n+k} = \frac{1}{(1-x)} \sum_k \binom{m}{k} \left(\frac{2x}{1-x}\right)^k$$

$$\Rightarrow \sum_k \binom{m}{k} x^{-k} \frac{x^m}{(1-x)^{m+1}} = \frac{1}{1-x}\left(1 + \frac{2x}{1-x}\right)^m$$

$$= \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{m}{k} x^{-k} \qquad = \frac{1}{1-x}\left(\frac{1+x}{1-x}\right)^m = \frac{(1+x)^m}{(1-x)^{m+1}}$$

$$= \frac{x^m}{(1-x)^{m+1}} \left(1 + \frac{1}{x}\right)^m.$$

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$$F(x) = \sum_{n \geq 0} \frac{x^n}{n!} f(n) = \sum_{n \geq 0} \frac{x^n}{n!} \sum_k \left[{n \atop k}\right] B_k$$

$$= \sum_k B_k \sum_{n \geq 0} \left[{n \atop k}\right] \frac{x^n}{n!}$$

$$= \sum_k B_k \left\{ \frac{1}{k!} \left(\log \frac{1}{1-x} \right)^k \right\}$$

$$= \sum_k \frac{B_k}{k!} u^k \qquad u = \log\left(\frac{1}{1-x}\right)$$

$$= \frac{u}{e^u - 1} = \frac{\log\left(\frac{1}{1-x}\right)}{\frac{1}{1-x} - 1} = \frac{(1-x)}{(1-(1-x))} \log\left(\frac{1}{1-x}\right)$$

$$= \frac{1-x}{x} \log\left(\frac{1}{1-x}\right)$$

$$F(n+1,k) - F(n,k) =$$

$$\sum_{k=-L}^{+L} \{F(n+1,k) - F(n,k)\} = \sum_{k=-L}^{+L} \{G(n,k+1) - G(n,k)\}$$

$$= (G(n,L+1) - \cancel{G(n,L)}) + (\cancel{G(n,L)} + G(n,\cancel{L-1})) + \cancel{\cdot}\ .$$

$$+ \cancel{\quad}$$

$$+(G(n,\cancel{-L+1}+1) - G(n,\cancel{-L+1})) + (G(n,\cancel{-L+1}) - G(n,-L))$$

$$= G(n,L+1) - G(n,-L)$$

$$F(n,k) = \frac{\binom{n}{k}}{2^n} \qquad \text{gives} \qquad G(n,k) = -\frac{\binom{n}{k-1}}{2^{n+1}}$$

$$F(n+1,k) - F(n,k) = \frac{\binom{n+1}{k}}{2^{n+1}} - \frac{\binom{n}{k}}{2^n}$$

$$\stackrel{?}{=} -\frac{\binom{n}{k}}{2^{n+1}} + \frac{\binom{n}{k-1}}{2^{n+1}}$$

Multiply by 2^{n+1}

$$\binom{n+1}{k} - 2\binom{n}{k} \stackrel{?}{=} -\binom{n}{k} + \binom{n}{k-1}$$

$$\|$$

$$\cancel{\binom{n}{k-1} + 2\binom{n-1}{k-1}}$$

$$\left\{ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \right\}$$

$$\binom{n}{k-1} + \binom{n}{k} - 2\binom{n}{k} \stackrel{?}{=}$$

✓ Yes

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$$\sum_k (-1)^k \binom{n}{k}\binom{2k}{k} 4^{n-k} = \binom{2n}{n}$$

$$\Leftrightarrow \sum_k \frac{(-1)^k \binom{n}{k}\binom{2k}{k} 4^{n-k}}{\binom{2n}{n}} = 1 \quad ✓$$

define $F(n,k) = \dfrac{(-1)^k \binom{n}{k}\binom{2k}{k} 4^{n-k}}{\binom{2n}{n}}$ ✓

defn $G(n,k) = R(n,k) \dfrac{(-1)^{k-1} \binom{n}{k-1}\binom{2k-2}{k-1} 4^{n-(k-1)}}{\binom{2n}{n}}$ ✓

condition 4.4.3 is

$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$

$(-1)^k($

Using definition of $R(n,k)$ given:

$$G(n,k) = \frac{(2k-1)}{(2n+1)} \frac{(-1)^{k-1} \binom{n}{k-1}\binom{2k-2}{k-1} 4^{n-(k-1)}}{\binom{2n}{n}}$$

Check

$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$

LHS =

$$\frac{(-1)^k\binom{n+1}{k}\binom{2k}{k}4^{n+1-k}}{\binom{2n+2}{n+1}} - \frac{(-1)^k\binom{n}{k}\binom{2k}{k}4^{n-k}}{\binom{2n}{n}} \quad \checkmark$$

$$= (-1)^k\binom{2k}{k}4^{n-k}\left\{\frac{\binom{n+1}{k}4^1}{\binom{2n+2}{n+1}} - \frac{\binom{n}{k}}{\binom{2n}{n}}\right\} \quad \checkmark$$

$$\stackrel{?}{=} RHS = (-1)^k\left(\frac{2k+1}{2n+1}\right)\binom{n}{k}\binom{2k}{k}\frac{4^{n-k}}{\binom{2n}{n}}$$

$$- (-1)^{k-1}\left(\frac{2k-1}{2n+1}\right)\binom{n}{k-1}\binom{2k-2}{k-1}\frac{4^{n-k+1}}{\binom{2n}{n}} \quad \checkmark$$

$$= \frac{(-1)^k 4^{n-k}}{\binom{2n}{n}}\left\{\left(\frac{2k+1}{2n+1}\right)\binom{n}{k}\binom{2k}{k} + \left(\frac{2k-1}{2n+1}\right)\binom{n}{k-1}\binom{2k-2}{k-1}\right\} \quad \checkmark$$

LHS =

$$\rightarrow (-1)^k\binom{2k}{k}4^{n-k}\left\{\frac{\frac{(n+1)!}{(n+1-k)!\,k!}(4)}{\frac{(2n+2)(2n+1)(2n)!}{(n+1)!\,(n+1)!}} - \frac{\binom{n}{k}}{\binom{2n}{n}}\right\}$$

$$= (-1)^k\binom{2k}{k}4^{n-k}\left\{\frac{\frac{(n+1)\,n!}{(n+1-k)(n-k)!\,k!}\,4}{\frac{(2n+2)(2n+1)}{(n+1)^2}\,\frac{(2n)!}{n!\,n!}} - \frac{\binom{n}{k}}{\binom{2n}{n}}\right\}$$

$$= \cancel{(-1)^k \binom{2k}{k} 4^{n-k}} \Bigg\{$$

$$= \frac{(-1)^k \binom{2k}{k} 4^{n-k} \binom{n}{k}}{\binom{2n}{n}} \left\{ \frac{\frac{4(n+1)}{n+1-k}}{\frac{2(2n+1)}{n+1}} - 1 \right\}$$

$$= \frac{(-1)^k \binom{2k}{k} 4^{n-k} \binom{n}{k}}{\binom{2n}{n}} \left\{ \frac{4(n+1)}{(n+1-k)} \cdot \frac{n+1}{2(2n+1)} - 1 \right\}$$

$$= \frac{(-1)^k \binom{2k}{k} 4^{n-k} \binom{n}{k}}{\binom{2n}{n}} \left\{ \frac{2(n+1)^2}{(n+1-k)(2n+1)} - \frac{(n+1-k)(2n+1)}{(n+1-k)(2n+1)} \right\}$$

$$= \cancel{\frac{(-1)^k \binom{2k}{k} 4^{n-k} \binom{n}{k}}{\binom{2n}{n}} \Bigg\{ 2(}$$

While RHS is

$$\frac{(-1)^k 4^{n-k}}{\binom{2n}{n}(2n+1)} \left\{ (2k+1)\binom{n}{k}\binom{2k}{k} + (2k-1)\binom{n}{k-1}\binom{2k-2}{k-1} \right\}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k \quad k(k-1)!}$$

$(n-k)\,(n-k-1)! \qquad n-(k+1) \qquad n-(k-1) = n-k+1$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$= \frac{n(n-1)!}{(n-k)(n}$$

$$\binom{n}{k-1} = \frac{n!}{(n-k+1)!\,(k-1)!} = \frac{n!\;k}{(n-k+1)(n-k)!\;k!} = \frac{k}{n-k+1}\binom{n}{k}$$

$$\binom{2k-2}{k-1} = \frac{(2k-2)!}{\underbrace{(2k-2-k+1)!}\,(k-1)!} \cdot \frac{k\,(2k-k)}{k\,(2k-k)}\,\frac{(2k-1)(2k)}{(2k-1)(2k)}$$

$$2k-k-1$$

$$= \frac{(2k)!}{k!\;k!} \cdot \frac{k(2k-k)}{(2k-1)(2k)} = \binom{2k}{k}\frac{k^2}{2k(2k-1)}$$

$$= \binom{2k}{k}\frac{k}{2(2k-1)}$$

$\therefore$ RHS

$$= \frac{(-1)^k\,4^{n-k}}{\binom{2n}{n}(2n+1)}\binom{n}{k}\binom{2k}{k}\left\{(2k+1) + \frac{(2k-1)k}{(n-k+1)}\,\frac{k}{2(2k-1)}\right\}$$

$$= \frac{(-1)^k\,4^{n-k}}{\binom{2n}{n}(2n+1)}\binom{n}{k}\binom{2k}{k}\left\{\frac{(2k+1)}{}\right.$$

Check . . .

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$$\sum_k \frac{(-1)^k \binom{n}{k}}{\binom{k+a}{k}} = \frac{a}{n+a}$$

$$\Leftrightarrow \sum_k \frac{(-1)^k \binom{n}{k}(n+a)}{\binom{k+a}{k} a} = 1$$

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$$f(x) = \prod_{i=1}^{n}(x-x_i) \qquad \ln f(x) = \sum_{i=1}^{n}\ln(x-x_i)$$

$\cancel{f(x)}$

$\frac{d}{dx}$

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{n}\frac{1}{x-x_i}$$

$$f'(x) = \sum_{i=1}^{n}\frac{f(x)}{x-x_i} = \cancel{\prod}\ \cancel{\sum(x-x_1)(x-x_2)\cdots(x-x_n)}$$

$$= (x-x_2)(x-x_3)\cdots(x-x_n) + (x-x_1)(x-x_3)\cdots(x-x_n) + (x-x_1)(x-x_2)(x-x_4)\cdots()$$

$$+ \cdots + (x-x_1)(x-x_2)\cdots(x-x_{n-1})$$

$$\frac{c_{n-m-2}(m+2)}{n-m-1}x^2 + 2c_{n-m-1}xy + \frac{(n-m)c_{n-m}}{m+1}y^2$$

$$\binom{n}{n-m-2}$$

$$A_n(y) = \{y(1+D_y)\}\, A_{n-1}(y) \qquad n \geq 0 \quad A_0 = 1$$

$$e^y A_n(y) = e^y y\left(A_{n-1} + A_{n-1}'(y)\right)$$

$$= y\left(e^y A_{n-1} + e^y A_{n-1}'\right)$$

$$= y\left(e^y A_{n-1}\right)'$$

0　identity mapping

$\binom{n}{2}$

$$\{b(n,k)\}_{k\geq 0} \xrightarrow{ops} (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1})$$

$\left\{ n \atop k \right\}$　Stirling #'s of 2nd kind
count set partitions.

$$\sum_{i=1}^{k} \sum_{t \geq 0} x^{a_i + b_i t}$$

$$= \sum_{i=1}^{k} x^{a_i} \sum_{t \geq 0} (x^{b_i})^t = \sum_{i=1}^{k} x^{a_i} \left(\frac{1}{1-x^{b_i}} \right) = \frac{1}{1-x}$$

$$\Rightarrow \sum_{i=1}^{k} \frac{x^{a_i}(1-x)}{(1-x^{b_i})} = 1$$

let $x \to 1$ $\Rightarrow$ $\cancel{\sum_{i=1}^{k} x^{a_i} (1-x)}$ $\sum_{i=1}^{k} \frac{x^{a_i}(-1)}{(-b_i x^{b_i - 1})} = 1$

By ~~to~~ L-Hopitls Rule

$$\Rightarrow \sum_{i=1}^{k} \frac{1}{b_i} = 1$$

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03-05-03

1

(1) $p_1 = p$

$p_2 = q \cdot p$ $\qquad q = 1-p$

$p_3 = q^2 p$

$\vdots$

$p_n = q^{n-1} p$

$$P(x) = \sum_{n\geq 1} p_n x^n = \sum_{n\geq 1} p q^{n-1} x^n = \left(\frac{p}{q}\right)\sum_{n\geq 1} q^n x^n = \left(\frac{p}{q}\right)\left[\frac{1}{1-qx} - 1\right]$$

$$= \left(\frac{p}{q}\right)\left[\frac{1-(1-qx)}{1-qx}\right] = \left(\frac{p}{q}\right)\left[\frac{qx}{1-qx}\right] = \frac{px}{1-qx}$$

Then $P(1) = \dfrac{p}{1-q} = 1$ ✓

$$\mu = \left.\frac{P'(x)}{P(x)}\right|_{x=1} = P'(1) = \left.\frac{p}{1-qx} - \frac{px}{(1-qx)^2}(-q)\right|_{x=1}$$

$$= 1 + \frac{pq}{p^2} = 1 + \frac{q}{p}$$

$$= 1 + \frac{1-p}{p} = \frac{1}{p}$$ ✓

$$\sigma^2 = \left\{ (\log P)' + (\log P)'' \right\}_{x=1}$$

$$\ln(P(x)) = \ln(px) - \ln(1-qx)$$

$$= \ln(x) - \ln(1-qx) + \ln p$$

$$(\ln(P(x)))' = \frac{1}{x} - \frac{(-q)}{1-qx}\Bigg|_{x=1} = 1 - \frac{-q}{1-q} = 1 + \frac{q}{p}$$

$$(\ln P(x))''\Bigg|_{x=1} = \frac{-1}{x^2} + \frac{q(-q)}{(1-qx)^2}\Bigg|_{x=1} = -1 - \frac{q^2}{p^2} = -\left(1 + \left(\frac{q}{p}\right)^2\right)$$

$$\therefore \quad \sigma^2 = 1 + \frac{q}{p} - 1 - \left(\frac{q}{p}\right)^2$$

$$= \frac{q}{p}\left(1 - \frac{q}{p}\right) = \left(\frac{1-p}{p}\right)\left(1 - \frac{(1-p)}{p}\right)$$

$$= \left(\frac{1}{p} - 1\right)\left(1 - \frac{1}{p} + 1\right)$$

positive somewhere

$$= \left(\frac{1-p}{p}\right)\left(\frac{p-1+p}{p}\right)$$

(2) d diff ~~photo~~ ~~photos~~ photos

$p_n =$ exactly n trials are needed to have a complete set

$p_n = 0$ if $n < d$

p_d ?

$\left\{ n \atop k \right\} =$ How many partitions of $[n]$ into k classes there are

$[4] = \{1, 2, 3, 4\}$

$\left\{ 4 \atop 2 \right\} = \cancel{7}\ 7$

If n trials are needed to obtain d stamps then on the n-th trial we get a complete set.

(a) ? $$p_n = \frac{d! \left\{ n-1 \atop d-1 \right\}}{d^n}$$

(b) $$P(x) = \sum_{n \geq d} p_n x^n = d! \sum_{n \geq d} \left\{ n-1 \atop d-1 \right\} \left(\frac{x}{d}\right)^n$$

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}$$

$$P(x) = d! \sum_{n \geq d} \left[\left\{ {n-2 \atop d-2} \right\} + (d-1) \left\{ {n-2 \atop d-1} \right\} \right] \left(\frac{x}{d}\right)^n$$

$$= d! \sum_{n \geq d} \left\{ {n-2 \atop d-2} \right\} \left(\frac{x}{d}\right)^n + d!(d-1) \sum_{n \geq d} \left\{ {n-2 \atop d-1} \right\} \left(\frac{x}{d}\right)^n$$

?

(c) ~~a~~ =

Check $P(1) = 1$?

$$P(1) = \frac{(d-1)!}{(d-1)(d-2)\cdots(d-(d-1))} = 1 \checkmark$$

$$\mu = \frac{P'(1)}{P(1)}$$

$$\cancel{P(x) = \frac{d!\, x^{d-1}}{(d-x)(d-2x)\cdots}}$$

$$\ln P(x) = \ln((d-1)!\, x^d) - \sum_{k=1}^{d-1} \ln(d-kx)$$

$$= \ln((d-1)!) + d \ln x - \sum_{k=1}^{d-1} \ln(d-kx)$$

$$\frac{d}{dx}(\ln P(x)) = \frac{P'(x)}{P(x)} = \frac{d}{x} - \sum_{k=1}^{d-1} \frac{(-k)}{d-kx}$$

$$= \frac{d}{x} + \sum_{k=1}^{d-1} \frac{k}{d-kx}$$

$$(\ln P(x))'\Big|_{x=1} = d + \sum_{k=1}^{d-1} \frac{k}{d-k} = d + \sum_{k=1}^{d-1} \frac{k-d+d}{-(k-d)}$$

$$= d - \sum_{k=1}^{d-1} \frac{k-d+d}{k-d} = d - \sum_{k=1}^{d-1} \left(1 + \frac{d}{k-d}\right)$$

$$= d - (d-1) - d \sum_{k=1}^{d-1} \frac{1}{k-d}$$

$$= \cancel{\boxed{k-d}} \sum$$

$$= 1 - d\left[\frac{1}{1-d} + \frac{1}{2-d} + \frac{1}{3-d} + \cdots + \frac{1}{-1}\right]$$

$$= 1 + d\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{d-1}\right]$$

$$= 1 + dH_{d-1} = d\left[1 + \frac{1}{2} + \cdots + \frac{1}{d-1} + \frac{1}{d}\right]$$

Absorb the 1

$$= dH_d$$

$$(\ln P(x))'' = \frac{d}{dx}\left[\frac{d}{x} + \sum_{k=1}^{d-1} \frac{k}{d-kx}\right]$$

$$= -\frac{d}{x^2} + \sum_{k=1}^{d-1} \frac{k(+k)}{(d-kx)^2}$$

$$= -\frac{d}{x^2} + \sum_{k=1}^{d-1} \frac{k^2}{(d-kx)^2}$$

$$(\ln P(x))''\Big|_{x=1} = -d + \sum_{k=1}^{d-1} \frac{k^2}{(d-k)^2}$$

$$= -d + \sum_{k=1}^{d-1} \frac{k^2}{k^2-2kd+d^2}$$

$$= -d + \sum_{k=1}^{d-1} \frac{k^2-2kd+d^2+2kd-d^2}{k^2-2kd+d^2}$$

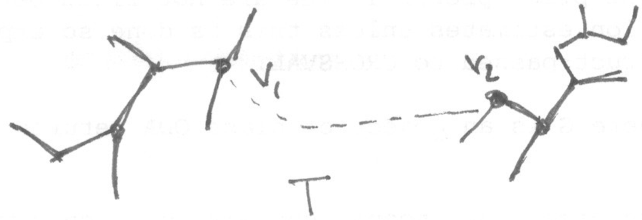
$$= -d + \sum_{k=1}^{d-1} 1 + \sum_{k=1}^{d-1} \frac{2kd-d^2}{(d-k)^2}$$

(e) $d = 10$

$$\mu = dH_d = 10\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right]$$

$$\cong 29.28$$

③ $p(j;v;T)$ … $\underline{T_1}$: $\underline{T_2}$:



$$F(x;v_1) = \sum_{j\geq 0} p(j;v_1;T_1)x^j$$

$$F(x;v_2) = \sum_{j\geq 0} p(j;v_2;T_2)x^j$$

(a) ?

(b) $F(x;v_1) = \dfrac{d_1}{d_1+1} F(x;v_1) + \dfrac{x^2}{(d_1+1)(d_2+1-d_2F(x;v_2))}$

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(4) $$N(x) = \sum_m e_{sm} x^m$$

$$\frac{E(x)}{1-x} = \frac{N(x-1)}{\cancel{x}-x}$$

(6) ?

(7) ?

(8)

Pr: $\sum_r \binom{n}{\lfloor \frac{r}{2} \rfloor} x^r = (1+x)(1+x^2)^n$ Apply snake oil

$$\sum_{n\geq 0} x^n \sum_r \binom{n}{\lfloor \frac{r}{2} \rfloor} x^r = \sum_r x^r \underbrace{\sum_{n\geq 0} \binom{n}{\lfloor \frac{r}{2} \rfloor} x^n}_{\frac{x^{\lfloor \frac{r}{2} \rfloor}}{(1-x)^{\lfloor \frac{r}{2} \rfloor + 1}}}$$

✓

$$= \sum_r \frac{x^{r + \lfloor \frac{r}{2} \rfloor}}{(1-x)^{\lfloor \frac{r}{2} \rfloor + 1}}$$ ✓

$$= \frac{1}{(1-x)} \sum_r x^r \left(\frac{x}{1-x} \right)^{\lfloor \frac{r}{2} \rfloor}$$ ✓

$$= \frac{1}{(1-x)} \Big[x^0 (\quad)^0 + x^1 (\quad)^0 + x^2 (\quad)^2$$

$$+ x^3 (\quad)^2 + \cdots$$

$$= \frac{1}{1-x}\left[\sum_{r \text{ even}} x^r \left(\frac{x}{1-x}\right)^{\lfloor \frac{r}{2} \rfloor} + \sum_{r \text{ odd}} x^r \left(\frac{x}{1-x}\right)^{\lfloor \frac{r}{2} \rfloor} \right] \checkmark$$

$$= \frac{1}{1-x}\left[\sum_{\substack{\cancel{r \text{ even}} \\ r \geq 0}} x^{2r} \left(\frac{x}{1-x}\right)^{r} + \sum_{r \geq 0} x^{2r+1} \left(\frac{x}{1-x}\right)^{r} \right] \checkmark$$

$$= \frac{1}{1-x}\left[\sum_{r \geq 0} \left(\frac{x^3}{1-x}\right)^{r} + x\sum_{r \geq 0} \left(\frac{x^3}{1-x}\right)^{r} \right] \checkmark$$

$$= \left[\frac{(1+x)}{1-x}\right] \sum_{r \geq 0} \left(\frac{x^3}{1-x}\right)^{r}$$

$$= \frac{(1+x)}{(1-x)} \frac{1}{1-\left(\frac{x^3}{1-x}\right)}$$

$$= \frac{1+x}{1-x-x^3}$$ same $\checkmark$

$$= \frac{1}{1-x}\left[\frac{1}{1-\frac{x^3}{1-x}} + \frac{x}{1-\frac{x^3}{1-x}} \right]$$

$$= \frac{1}{1-x}\left[\frac{1+x}{1-\frac{x^3}{1-x}} \right]$$

$$= \frac{1+x}{1-x-x^3} = \frac{1+x}{1-x(1+x^2)} = (1+x)\sum_{n \geq 0} x^n (1+x^2)^n$$

$$= (1+x)\sum_{n \geq 0} x^n \sum_{k \geq 0} \binom{n}{k} x^{2k}$$

$$= (1+x)\sum_{k \geq 0} \sum_{n \geq 0} \binom{n}{k} x^{n+2k}$$

$$\cancel{= (1+x)\sum_{k \geq 0} x^{2k} \underbrace{\sum_{n \geq 0} \binom{n}{k} x^{n}}_{\frac{x^k}{(1-x)^{k+1}}}}$$

$\therefore$ ~~$\sum$~~ $\qquad A_n \equiv \sum_r \binom{n}{\lfloor \frac{r}{2} \rfloor} x^r$

$$A(x) \equiv \sum_{n \geq 0} A_n x^n = \frac{1+x}{1-x-x^3} = \frac{1+x}{1-x(1+x^2)}$$

$$A_n = \frac{1}{n!} \frac{d^n}{dx^n} A(x) \Big|_{x=0}$$

$$\frac{dA}{dx} = \frac{1}{1-x-x^3} + \frac{-(1+x)(-1-3x^2)}{(1-x-x^3)^2}$$

$$= \frac{1}{1-x-x^3} + \frac{(1+x)(1+3x^2)}{(1-x-x^3)^2} = \frac{1-\cancel{x}-x^3+1+3x^2+\cancel{x}+3x^3}{(1-x-x^3)^2}$$

$$= \frac{2+3x^2+2x^3}{(1-x-x^3)^2}$$

Since OPSGF of L.H.S is $A(x)$ above what is O.P.S.G.F. of right hand side

$$(1+x)\sum_{n \geq 0} x^n (1+x^2)^n = \frac{(1+x)}{1-x(1+x^2)}$$ same ✓. equality holds

consider

$$\sum_k \binom{n}{k} \binom{n-k}{\lfloor \frac{m-k}{2} \rfloor} y^k \Big|_{y=2} = \sum_k \binom{n}{k} \binom{n-k}{\lfloor \frac{m-k}{2} \rfloor} 2^k$$

Note: an alternative method is to compute the roots of $1-x-x^3 \cong (x-x_1)(x-x_2)(x-x_3)$

& to a partial fraction expansion of the rational fns

i.e. $$\frac{1+x}{1-x-x^3} = \frac{1+x}{(x-a)(x-b)(x-c)} = (1+x)\left[\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}\right]$$

each term ~~can~~ can then be written as $\sum_{r\geq 0} x^r$.

(11) $\sum_{k\geq 0}\binom{k}{n-k}t^k$

$$\sum_{n\geq 0}x^n\sum_{k\geq 0}\binom{k}{n-k}t^k = \sum_{k\geq 0} t^k\sum_{n\geq 0}\binom{k}{n-k}x^n$$

$$= \sum_{k\geq 0} t^k x^k \underbrace{\sum_{n\geq 0}\binom{k}{n-k}x^{n-k}}_{(1+x)^k} = \sum_{k\geq 0}[x(1+x)t]^k$$

$$= \frac{1}{1-xt(1+x)} = \frac{1}{1-tx-tx^2} = \frac{A}{1-x} =$$

$$1-tx-tx^2 = 0$$

$$\Rightarrow x = \frac{t\pm\sqrt{t^2+4t}}{2(-t)}$$

$$= \frac{-1}{t(x^2+x-1/t)}$$

$$= \frac{-1}{t}\cdot\frac{1}{(x^2+x-1/t)}$$

roots of x^2+x-1/t or $x_r = \frac{-1\pm\sqrt{1+4/t}}{2}$

$$= \frac{-1}{t}\frac{1}{(x-x_-)(x-x_+)}$$

Now

$$\frac{1}{(x-x_-)(x-x_+)} = \frac{A}{x-x_-} + \frac{B}{x-x_+}$$

w/

$$A = \frac{1}{x_- - x_+} \qquad B = \frac{1}{x_+ - x_-} = -A$$

$$= \cancel{\frac{1}{\frac{-1-\sqrt{1+4/t}}{2} -}}$$

$$= \frac{1}{\left(\frac{-1-\sqrt{1+4/t}}{2}\right) - \left(\frac{-1+\sqrt{1+4/t}}{2}\right)} = \frac{2}{-2\sqrt{1+4/t}} = \frac{-1}{\sqrt{1+4/t}}$$

$\therefore$

$$\frac{1}{1-xt(1+x)} = \frac{-1}{t}\left[\frac{A}{x-x_-} + \frac{B}{x-x_+}\right]$$

$$= \frac{-1}{t}\left[\frac{-A}{x_-(1-x/x_-)} + \frac{-B}{x_+(1-x/x_+)}\right]$$

(c)

$$\sum_k \binom{2n+1}{2p+2k+1}\binom{p+k}{k}$$

$$\sum_{n\geq 0} x^{2n} \sum_k \binom{2n+1}{2p+2k+1}\binom{p+k}{k}$$

$$= \sum_k \binom{p+k}{k} \sum_{n\geq 0} x^{2n}\binom{2n+1}{2p+2k+1}$$

$$= \sum_k \binom{p+k}{k} x^{-1} \underbrace{\sum_{n\geq 0} x^{2n+1}\binom{2n+1}{2p+2k+1}}_{\frac{x^{2p+2k+1}}{(1-x)^{2p+2k+2}}}$$

$$= \sum_k \binom{p+k}{k} \frac{x^{2p+2k}}{(1-x)^{2p+2k+2}} = \left[\frac{x^2}{(1-x)^2}\right]^p \left(\underbrace{\sum_k \binom{p+k}{k} \frac{x^{2k}}{(1-x)^{2k}}}\right)\frac{1}{(1-x)^2}$$

consider $\displaystyle\sum_k \binom{p+k}{k}\frac{x^{2k}}{(1-x)^{2k}}$ mult by t^p + sum over p to evaluate this sum

$$\sum_k \frac{x^{2k}}{(1-x)^{2k}} \underbrace{\sum_p \binom{p+k}{k} t^p}$$

$$\underbrace{t^{-k} \sum_p \binom{p+k}{k} t^{p+k}}$$

$$t^{-k} \cdot \frac{t^k}{(1-t)^{k+1}} = \frac{1}{(1-t)^{k+1}}$$

$$\sum_k \frac{x^{2k}}{(1-x)^{2k}} \cdot \frac{1}{(1-t)^{k+1}} = \frac{1}{1-t} \sum_k \left[\frac{x^2}{(1-x)^2(1-t)} \right]^k$$

$$= \frac{1}{1-t} \frac{1}{1 - \frac{x^2}{(1-x)^2(1-t)}} = \frac{1}{\cancel{(1-t)}} \cdot \frac{(1-x)^2\cancel{(1-t)}}{(1-x)^2(1-t) - x^2}$$

$$= \frac{(1-x)^2}{(1-x)^2(1-t) - x^2} = \frac{(1-x)^2}{(1-x)^2 - x^2 - (1-x)^2 t} = \frac{(1-x)^2}{}$$

...

Algebra gets a crazy.

Consider the sum $\sum_k \binom{p+k}{k} \cancel{x}^k$

w/ "x" = $\frac{x^2}{(1-x)^2}$ mult by t^p + sum

$$\sum_k x^k \underbrace{\sum_p \binom{p+k}{k} t^p}$$

$$\underbrace{t^{-k} \sum_p \binom{p+k}{k} t^{p+k}}$$

$$\cdot \frac{t^k}{(1-t)^{k+1}} = \frac{1}{(1-t)^{k+1}}$$

$$= \sum_k \left(\frac{x}{1-t}\right)^k \cdot \frac{1}{(1-t)}$$

$$= \frac{1}{1-t} \frac{1}{\left(1-\left(\frac{x}{1-t}\right)\right)} = \frac{1}{1-t-x} = \frac{1}{1-x} \frac{1}{1-\frac{t}{(1-x)}}$$

$$= \frac{1}{1-x} \sum_p \frac{t^p}{(1-x)^p}$$

$$\sum_k \binom{p+k}{k} x^k = \frac{1}{(1-x)^{p+1}}$$

Thus sum from before

$$\frac{x^{2p}}{(1-x)^{2p+2}} \cdot \frac{1}{\left(1-\frac{x^2}{(1-x)^2}\right)^{p+1}} = \frac{x^{2p}}{\cancel{(1-x)^{2p+2}}} \; \frac{\cancel{(1-x)^{2(p+1)}}}{\left((1-x)^2 - x^2\right)^{p+1}}$$

$$= \frac{x^{2p}}{(1-2x)^{p+1}} = \frac{x^{2p}}{1-2x} \quad \cancel{(1-2x)^{p}} \; \frac{x^{2p}}{\cancel{(1-2x)^{2p+1}}}$$

$$= \cancel{(1-2x)^{p}}$$

$$= \cancel{x^{p}} \; \frac{(2x)^{2p}}{2^{2p}\,(1-2x)^{p+1}} = \frac{(2x)^{p}}{2^{2p}} \; \frac{(2x)^{p}}{(1-2x)^{p+1}}$$

$$= \frac{x^p}{2^p} \sum_{n \geq 0} \binom{n}{p} (2x)^n = \frac{x^p}{2^p} \sum_{n \geq 0} \binom{n}{p} 2^n x^n$$

(d) $$\sum_m \binom{r}{m}\binom{s}{t-m} = \binom{r+s}{t}$$

To answer second part pick ~~t~~ $r = n$

$$\sum_k \binom{n}{k}\binom{s}{t-k} = \binom{n+s}{t}$$

~~S~~ t. $\binom{n}{k} = \binom{n}{n-k}$ so pick $t \;\&\; s = n$

$$\Rightarrow \sum_k \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

To answer 1st part:

$$\sum_m \binom{r}{m}\binom{s}{t-m} = \;?$$

multiply by x^t & sum over t.

$$\cancel{\sum_m \binom{s}{t-m} \sum} \qquad \sum_m \binom{r}{m} \sum_t \binom{s}{t-m} x^t$$

$$\sum_m \binom{r}{m} x^m \underbrace{\sum_t \binom{s}{t-m} x^{t-m}}_{(1+x)^s}$$

$$= (1+x)^s (1+x)^r = (1+x)^{s+r} = \sum_k \binom{s+r}{t} x^t$$

$$\therefore \quad \sum_m \binom{r}{m}\binom{s}{t-m} = \binom{s+r}{t}$$

(e)
$$\sum_k \binom{2n+1}{2k}\binom{m+k}{2n} = \binom{2m+1}{2n}$$

multiply by x^m + sum over m.

$$\sum_{m\geq 0} x^m \sum_k \binom{2n+1}{2k}\binom{m+k}{2n} = \sum_k \binom{2n+1}{2k} \sum_{m\geq 0} x^m \binom{m+k}{2n}$$

$$= \sum_k \binom{2n+1}{2k} x^{-k} \underbrace{\sum_{m\geq 0} x^{m+k}\binom{m+k}{2n}}_{\frac{x^{2n}}{(1-x)^{2n+1}}} = \frac{x^{2n}}{(1-x)^{2n+1}} \sum_k \binom{2n+1}{2k} x^{-k}$$

$$\cancel{= \sum_k} = \frac{x^{2n}}{(1-x)^{2n+1}} \left[\text{How sum?} \right]$$

Rv:

(f)
$$\sum_k \binom{n}{k}\binom{k}{j} x^k = \binom{n}{j} x^j (1+x)^{n-j}$$

mult by y^j + sum over j

$$\sum_k \binom{n}{k} x^k \sum_j \binom{k}{j} y^j = \sum_k \binom{n}{k} x^k (1+y)^k$$

consider

$$\sum_k \binom{2n+1}{2k} x^{-k} = \sum_k \binom{2n+1}{2k} (\sqrt{x})^{-2k}$$

$$= (1+\sqrt{x})^{2n+1} \qquad \therefore \text{ L.H.S is}$$

$$\frac{x^{2n}}{(1-x)^{2n+1}} \cdot (1+\sqrt{x})^{2n+1} = \frac{x^{2n}}{(1-\sqrt{x})^{2n+1}}$$

$$= \cancel{x^{2n}} \qquad x^{2n}\,(1-\sqrt{x})^{-(2n+1)} \qquad \ldots ?$$

$$= (x(1+y)+1)^n = (1+x+xy)^n$$

$$= (1+x)^n \left[1 + \left(\frac{x}{1+x}\right)y\right]^n = (1+x)^n \sum_j \binom{n}{j}\left(\frac{x}{1+x}\right)^j y^j$$

$$\Rightarrow \sum_k \binom{n}{k}\binom{k}{j} x^k = (1+x)^n \binom{n}{j}\left(\frac{x}{1+x}\right)^j$$

$$= \binom{n}{j} x^j (1+x)^{n-j} \quad \checkmark$$

Pr:

(g) $$x \sum_k \binom{n+k}{2k}\left(\frac{x^2-1}{4}\right)^{n-k} = \left(\frac{x-1}{2}\right)^{2n+1} + \left(\frac{x+1}{2}\right)^{2n+1}$$

Mult by y^n + sum over n

$$x \sum_k \sum_{n \geq 0} y^n \binom{n+k}{2k}\left(\frac{x^2-1}{4}\right)^{n-k}$$

$$\Rightarrow x\left(\frac{x^2-1}{4}\right)^n \sum_k \binom{n+k}{2k}\left(\frac{x^2-1}{4}\right)^{-k} = \cdots$$

Pr: $$\sum_k \binom{n+k}{2k}\left(\frac{x^2-1}{4}\right)^{-k} = \left(\frac{4}{x^2-1}\right)^n \frac{1}{x}\left[\left(\frac{x-1}{2}\right)^{2n+1} + \left(\frac{x+1}{2}\right)^{2n+1}\right] \blacksquare$$

Thus L.H.S =

$$\sum_k \binom{n+k}{2k} \left(\frac{4}{x^2-1}\right)^k$$

consider $\sum_k \binom{n+k}{2k} x^k$ mult by y^n & sum over n

$$\sum_k x^k \sum_n \binom{n+k}{2k} y^n = \sum_k x^k y^{-k} \underbrace{\sum_n \binom{n+k}{2k} y^{n+k}}_{\frac{y^{2k}}{(1-y)^{2k+1}}}$$

$$= \sum_k x^k \frac{y^k}{(1-y)^{2k+1}} = \frac{1}{(1-y)} \sum_k \left(\frac{xy}{\sqrt{1-y}}\right)^k$$

$$= \frac{1}{(1-y)} \cdot \frac{1}{1 - \frac{xy}{\sqrt{1-y}}} \quad \ldots$$

(h)
$$\sum_{k\geq 1} \binom{n+k+1}{2k-1} \frac{(x-1)^{2k} x^{n-k}}{k} =$$

(12)
$$b_n = \sum_k \left[{n \atop k}\right] a_k$$

mult by $\frac{x^n}{n!}$ & sum over n

$$\Rightarrow \quad B(x) = \sum_n \frac{x^n}{n!} \sum_k \left[{n \atop k}\right] a_k = \sum_k a_k \underbrace{\sum_n \frac{x^n}{n!} \left[{n \atop k}\right]}_{\text{How sum?}}$$

(b)

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(13)

$$\sum_k (-1)^{n-k} \binom{2n}{k}^2 = \sum_k (-1)^{n-k} \binom{\cancel{2n+m}}{k}\binom{2m}{k}$$

(14)

$$S(n) = \sum_k f(k)\, g(n-k)$$

$S(n) = [x^n]\{F(x)G(x)\}$ w/ F, G as opsgf of $\{f_n\}$ + $\{g_n\}$

$$S(n) = \sum_k \frac{1}{k+1}\binom{2k}{k} \frac{1}{n-k+1}\binom{2n-2k}{n-k}$$

let $f_n = \frac{1}{n+1}\binom{2n}{n}$

Then op.s.g.f. of f_n = ?

$$F(x) = \sum_n \frac{1}{n+1}\binom{2n}{n} x^n$$

$$xF(x) = \sum_n \frac{1}{n+1}\binom{2n}{n} x^{n+1}$$

$$\frac{d}{dx}(xF) = \sum_n \binom{2n}{n} x^n = ?$$ How sum?

$$\frac{(2n)!}{n!\,n!} = \frac{(2n)(2n-1)(2n-2)\cdots(n+1)}{n!}$$

$$= \frac{2n(2n-2)(2n-4)\cdots}{n!}$$

15 Pr:

(a) $$\sum_r \binom{m}{r}\binom{n-r}{n-r-q}(t-1)^r = \sum_r \binom{m}{r}\binom{n-m}{n-r-q} t^r$$

(b)

16 $$f_n = \sum_k \binom{n+k}{m+2k} c_k$$

$$F(x) = \sum_k c_k \sum_n \binom{n+k}{m+2k} x^n = \sum_k c_k x^{-k} \underbrace{\sum_n \binom{n+k}{m+2k} x^{n+k}}_{\frac{x^{m+2k}}{(1-x)^{m+2k+1}}}$$

$$= \sum_k \frac{x^{m+k}}{(1-x)^{m+2k+1}} c_k$$

$$= \frac{x^m}{(1-x)^{m+1}} \underbrace{\sum_k \frac{x^k}{(1-x)^{2k}} c_k}$$

$$\frac{x^m}{(1-x)^{m+1}} C\left(\frac{x}{(1-x)^2}\right)$$

Say something special... was c_k finite?

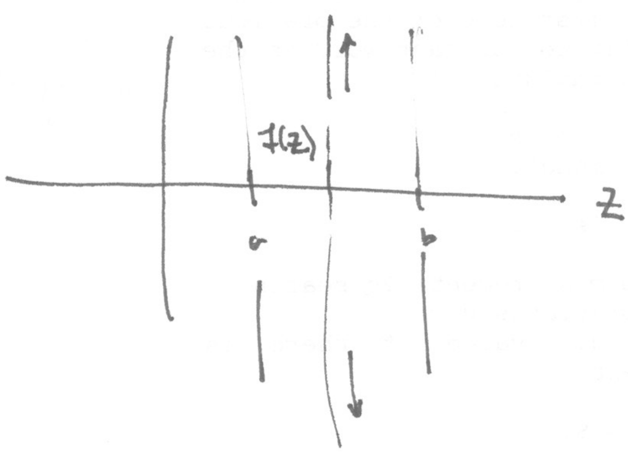
(17) (a) $F_x = G_y$ $\qquad \lim_{y\to\infty} G(x,y) = 0$

consider $I(x) \equiv \int_{-\infty}^{+\infty} F(x,y)\, dy$

$$I'(x) = \int_{-\infty}^{+\infty} F_x(x,y)\, dy = \int_{-\infty}^{+\infty} G_y(x,y)\, dy = G(x,y)\Big|_{-\infty}^{+\infty} = 0$$

$\Rightarrow$ $I(x)$ is a constant.

(b)



$f(z)$ analytic $\Rightarrow$ $f(z) = f(x+iy) = F(x,y) + iG(x,y)$

By Cauchy Riemann eqs analiticity of f

$\Rightarrow$ $\frac{\partial F}{\partial x} = \frac{\partial G}{\partial y}$; $\frac{\partial G}{\partial x} = -\frac{\partial F}{\partial y}$

on a vertical line $z = x_0 + i\xi$ $\qquad \xi \to \pm\infty$.

+ f vanishes $\Rightarrow$ F + G vanish

(c) ?

(d) $e^{z^2} = \cancel{e^{x^2+y^2+2}} \quad e^{x^2-y^2+2ixy}$

$$= e^{x^2-y^2}\left[\cos(2xy) + i\sin(2xy)\right]$$

$\therefore$
$$F(x,y) = e^{x^2-y^2}\cdot\cos(2xy)$$
$$G(x,y) = e^{x^2-y^2}\cdot\sin(2xy)$$

Then
$$\int_{-\infty}^{+\infty} e^{x^2-y^2}\cos(2xy)\,dy = C$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-y^2}\cos(2xy)\,dy = Ce^{-x^2}.$$

$C=?$ pix $x=0$

$$\int_{-\infty}^{+\infty} e^{-y^2}\,dy = \equiv I$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$= 2\pi\left(\frac{e^{-r^2}}{(-1)}\right)\Bigg|_0^{\infty} = 2\pi$$

$$I = \sqrt{2\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-y^2}\cos(2xy)\,dy = \sqrt{2\pi}\cdot e^{-x^2}.$$

(18)

(a) $d_1, \dots, d_n$

(b) $$F_n(x_1, \dots, x_n) = \sum_{\substack{d_1+d_2+\cdots+d_n = 2(n-1) \\ d_1, \dots, d_n \geq 1}} f_n(d_1, d_2, \dots, d_n)\, x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$$

$$= \sum_{d_1+d_2+\cdots+d_n = 2(n-1)} \frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!} x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$$

$$= \sum (n-2)! \frac{x_1^{d_1}}{(d_1-1)!} \frac{x_2^{d_2}}{(d_2-1)!} \cdots \frac{x_n^{d_n}}{(d_n-1)!}$$

...

$$= \sum^{(n-2)}$$

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$S \to S' \to S'' \cdots$

$f(n,k,r) = f(n,k+1,$

$= f($

(b) Avg # of steps $= \sum_{k} k f(n,k,0)$ *

(a) $f(n,k,r) = f(n,k+1,r-\tilde{r})$

$y = x + x^2$

$x^2 + x - y = 0$

$x = \frac{-1 \pm \sqrt{1-4(-y)}}{2} = \frac{-1 \pm \sqrt{1+4y}}{2}$

But $y = 0$ when $x = 0$

+ $x = 0$ when $y = 0$ ∴ take plus sign

Thm 2.4.3

$$|a_n| \le \left(\frac{1}{|z_0|}+\epsilon\right)^n \quad \exists N \ni \forall n \ge N$$

For ∞ many n we have

$$|a_n| > \left(\frac{1}{|z_0|}-\epsilon\right)^n$$

$$\tilde{b}(n) = \sum_{k=1}^{n} k! \left\{ n \atop k \right\}$$

eq 4.246 is

$$\sum_{1\le k\le n} \left\{ n \atop k \right\} y^k = e^{-y} \sum_{r\ge 1} \frac{r^n}{r!} y^r$$

$$\Rightarrow \sum_{1\le k\le n} \left\{ n \atop k \right\} \int_0^\infty y^k e^{-y}\,dy = \sum_{r\ge \cancel{0}1} \frac{r^n}{r!} \int_0^\infty e^{-2y} y^r\,dy$$

$$\Gamma(n+1) \equiv \int_0^\infty x^n e^{-x}\,dx = n! \quad n \text{ integer}$$

$$v = 2y \qquad dv = 2\,dy$$

$$\Rightarrow \sum_{1\le k\le n} \left\{ n \atop k \right\} k! = \sum_{r\ge 1} \frac{r^n}{r!} \int_0^\infty e^{-v} \frac{v^r}{2^r} \frac{dv}{2}$$

$$\tilde{b}(n) = \sum_{r\ge 1} \frac{r^n}{\cancel{r!}\, 2^{r+1}} \cancel{r!}$$

definition of ordered bell #'s.

$$= \frac{1}{2} \sum_{r\ge 1} \frac{r^n}{2^r}$$

$$\tilde{b}(n) = \sum_{r\geq 0} \frac{r^n}{2^{r+1}} = \cancel{\frac{1}{2}\sum_{r\geq 0}\left(\frac{r}{2}\right)^n} \quad \frac{1}{2}\sum_{r\geq 0}\frac{r^n}{2^r}$$

mult by $\frac{x^n}{n!}$ + sum over n

$$f(z) = \sum_{n\geq 0} \frac{\tilde{b}(n)}{n!} z^n = \frac{1}{2}\sum_{n\geq 0}\frac{1}{n!}\sum_{r\geq 0}\frac{r^n}{2^r} z^n$$

$$= \cancel{\frac{1}{2}\sum_{r\geq 0}\frac{1}{2^r}\underbrace{\sum_{n\geq 0} r^n}_{\frac{1}{1-r}}} = \frac{1}{2}\sum_{r\geq 0}\frac{1}{2^r}\underbrace{\sum_{n\geq 0}\frac{(rz)^n}{n!}}_{e^{rz}}$$

$$= \frac{1}{2}\sum_{r\geq 0}\left(\frac{e^z}{2}\right)^r$$

$$= \frac{1}{2}\,\frac{1}{1-\left(\frac{e^z}{2}\right)} = \frac{1}{2-e^z} \quad \text{eq 5.2.7}$$

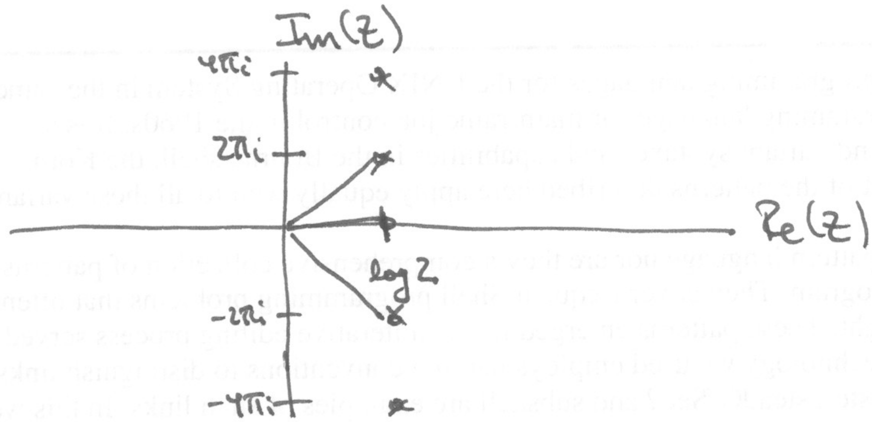
$$2 - e^z = 0$$

$$e^z = 2$$

$$z = \log 2 \pm i\,2\pi k$$ these are simple poles

$$e^z = e^{\log 2} \cdot e^{\pm 2\pi i k} \quad \checkmark$$

$$f(z) = \frac{1}{2 - e^z} = \frac{}{(z - \log 2} + \sum_{n \geq 0} c_n (z - z_0)^n$$



point closest to the origin is $z_0 = \ln 2$

$$2 - e^z = (z - z_0)\,\phi(z)$$

$$\phi(z) = \frac{2 - e^z}{(z - z_0)} = \frac{f(z)}{g(z)}$$

$$\lim_{z \to z_0} \phi(z) = \frac{f'(z_0)}{g'(z_0)} = f'(z_0) = -e^z \Big|_{z_0 = \ln 2} = -2$$

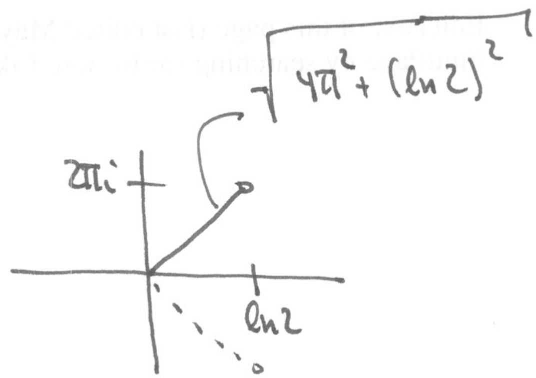
$$\therefore\ f(z) \equiv \frac{1}{2-e^{z}} = \frac{-1}{2}\,\frac{1}{(z-\ln 2)} + \sum_{n\geq 0} c_n (z-z_0)^n$$

$$= +\frac{1}{2}\,\frac{1}{\ln 2\,(1 - z/\ln 2)} + \sum_{n\geq 0} c_n (z-z_0)^n$$

$$= \frac{1}{2\ln 2} \sum_{n\geq 0} \frac{z^n}{(\ln 2)^n} + \text{''}$$

$$= \sum \frac{z^n}{2(\ln 2)^{n+1}} + \text{''}$$

$$h(z) = \frac{1}{2-e^{z}} - \frac{(-1/2)}{(z-\ln 2)}$$



coefficients of $h_n = O((.16)^n)$

Next closest poles to the origin are at $(\ln 2 \pm 2\pi i)$

The residue of these points is

$$a_{-1} = \lim_{z\to z_0} \frac{1}{(2-e^{z})'} = \lim_{z\to z_0} \frac{-1}{e^{z}} = \frac{-1}{e^{z_0}} = \frac{-1}{2}$$

define

$$g(z) \equiv \frac{1}{2-e^{z}} - \frac{(-1/2)}{(z-\ln 2)} - \frac{(-1/2)}{(z-(\ln 2 - 2\pi i))} - \frac{(-1/2)}{(z-(\ln 2 + 2\pi i))}$$

Now the o.p.s. coeff of $g(z)$ g_n are known $g_n = O\left(\left(\frac{1}{R}\right)^n\right)$ w/ R the Radius to the 1st singularity of $g(z)$

$$R = \sqrt{(\ln 2)^2 + (4\pi)^2} = 12.6$$

$$\Rightarrow |g_n| = O((.08)^n)$$

?. Now

$$g(z) = \frac{1}{2-e^z} - \frac{(+1/2)}{(1 - z/\ln 2)\ln 2}$$

$$- \frac{(+1/2)}{(\ln 2 - 2\pi i)\left(1 - \frac{z}{\ln 2 - 2\pi i}\right)}$$

$$- \frac{(+1/2)}{(\ln 2 + 2\pi i)\left(1 - \frac{z}{\ln 2 + 2\pi i}\right)}$$

$$= \frac{1}{2-e^z} - \frac{1}{2\ln 2}\sum_{n\geq 0}\frac{z^n}{(\ln 2)^n} - \frac{1}{2(\ln 2 - 2\pi i)}\sum_{n\geq 0}\frac{z^n}{(\ln 2 - 2\pi i)^n}$$

$$- \frac{1}{2(\ln 2 + 2\pi i)}\sum_{n\geq 0}\frac{z^n}{(\ln 2 + 2\pi i)^n}$$

$$= \frac{1}{2-e^z} - \frac{1}{2}\sum_{n\geq 0}\left(\frac{1}{(\ln 2)^{n+1}} + \frac{1}{(\ln 2 - 2\pi i)^{n+1}} + \frac{1}{(\ln 2 + 2\pi i)^{n+1}}\right) z^n$$

$$\Rightarrow \left| f_n - \frac{1}{2}\left[\frac{1}{(\ \ )^{n+1}} + \frac{1}{(\ \ )^{n+1}} + \frac{1}{(\ \ )^{n+1}}\right]\right| = O\left((.08)^n\right)$$

?

$$\Rightarrow \tilde{b}(n) = \cancel{\ \ }$$

$$\frac{1}{2}\left[\frac{1}{(\ln 2)^{n+1}} + \frac{1}{(\ln 2 - 2\pi i)^{n+1}} + \frac{1}{(\ln 2 + 2\pi i)^{n+1}}\right] n!$$

$$+ O\left(n!(.08)^n\right)$$

$$= \frac{1}{2}\left[\frac{1}{(\ln 2)^{n+1}} + \underbrace{\frac{1}{(\ln 2)^{n+1}\left(1 - \frac{2\pi i}{\ln 2}\right)^{n+1}} + \frac{1}{(\ln 2)^{n+1}\left(1 + \frac{2\pi i}{\ln 2}\right)^{n+1}}}\right] + \cdots$$

consider

$$\frac{1}{\left(1 - \frac{2\pi i}{\ln 2}\right)^{n+1}} + \frac{1}{\left(1 + \frac{2\pi i}{\ln 2}\right)^{n+1}}$$

$$\left\{ \frac{1}{(1-x)^{k+1}} = \sum_n \binom{n+k}{n} x^n \right\}$$

$$\Rightarrow \sum_k \binom{k+n}{k}\left(\frac{2\pi i}{\ln 2}\right)^k$$

$$+ \sum_k \binom{k+n}{k}\left(\frac{-2\pi i}{\ln 2}\right)^k = \sum_k \binom{k+n}{k} \frac{(2\pi)^k}{(\ln 2)^k}\left[i^k + (-i)^k\right]$$

$$i^k + (-i)^k = \begin{array}{ll} 0 & k=1 \\ -1-1=-2 & k=2 \\ -i+i=0 & k=3 \\ +1+1=2 & k=4 \\ \vdots & \end{array} = \begin{array}{ll} 0 & k \text{ odd} \\ (-1)^{k/2} & k \text{ even} \end{array}$$

How can I show that this sum is smaller than ~~the~~ than the previous one?

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$$h(z) = f_q(z) - \frac{e^{-H_q}}{1-z}$$

$$= \frac{1}{1-z} e^{-\{z + z^2/2 + z^3/3 + \cdots + z^q/q\}} - \frac{e^{-H_q}}{1-z}$$

$$= \frac{e^{-\{z + z^2/2 + z^3/3 + \cdots + z^q/q\}} - e^{-H_q}}{1-z}$$

n-th coeff of $h(z)$

$$[z^n](1-z)^{\beta} = \binom{\beta}{n}(-1)^n$$

$$= \left[\beta(\beta-1)(\beta-2)(\beta-3)\cdots(\beta-n+3)(\beta-n+2)(\beta-n+1)\right](-1)^n$$

$$= (n-\beta-1)(n-\beta-2)(n-\beta-3)\cdots(3-\beta)(2-\beta)(1-\beta)(-\beta)$$

$$= \binom{n-\beta-1}{n} = \frac{(n-\beta-1)!}{n!(-\beta-1)!}$$

$n-\beta-1-(\square-1)$ w/ $\square$ = Bottom of choose notation.

$n-\beta-1-\square+1$

$n-\beta-\square \equiv -\beta$

↑ set

$\square = n$.

$$\neq \cancel{\Gamma(n-\beta-1)}$$

$\{\Gamma(n+1) = n! \quad n \text{ integer}\}$

$$= \cancel{\Gamma(n-\beta-}\ \frac{\Gamma(n-\beta)}{\Gamma(n+1)\Gamma(-\beta)}$$

$\cancel{\Gamma n}\ n! \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$

$n \gg 1$

$$\sim \frac{\left(\frac{n-\beta}{e}\right)^{n-\beta}\sqrt{2\pi(n-\beta)}}{\left(\frac{n+1}{e}\right)^{n+1}\sqrt{2\pi(n+1)}\cdot\cancel{\left(\frac{-\beta}{e}\right)^{-\beta}\sqrt{2\pi(-\beta)}}\ \Gamma(-\beta)}$$

$$= \sqrt{\frac{n-\beta}{n+1}}\cdot\frac{\left(\frac{n-\beta}{e}\right)^n\left(\frac{n-\beta}{e}\right)^{-\beta}}{\left(\frac{n+1}{e}\right)^n\left(\frac{n+1}{e}\right)\ \Gamma(-\beta)}$$

large n limit

$$\Rightarrow \sim \frac{\cancel{n^{n}}\left(1-\beta/n\right)^{n}\left(\frac{n-\beta}{e}\right)^{-\beta}}{\cancel{n^{n}}\left(1+1/n\right)^{n}\left(\frac{n+1}{e}\right)\Gamma(-\beta)}$$

$$\sim \frac{\cancel{e^{-\beta}}\,(n-\beta)^{-\beta}\,\cancel{e^{+\beta}}}{\cancel{e^{1}}\left(\frac{n+1}{\cancel{e}}\right)\Gamma(-\beta)} \qquad n \to +\infty$$

$$= \frac{(n-\beta)^{-\beta}}{(n+1)\,\Gamma(-\beta)} \simeq \frac{n^{-\beta}\left(1-\beta/n\right)^{-\beta}}{n\left(1+1/n\right)\Gamma(-\beta)}$$

$$\sim \frac{n^{-\beta-1}}{\Gamma(-\beta)} \qquad n \to +\infty$$

eq 8.3.3

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$$[z^n](1-z)^{\beta+j} = \binom{n-\beta-j-1}{n}$$

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$$\min_{r>0} \frac{e^r}{r^n}$$

$$\frac{d}{dr}\left(\frac{e^r}{r^n}\right) = \frac{e^r}{r^n} - n\frac{e^r}{r^{n+1}} \underset{\text{set}}{=} 0$$

$$= \frac{1}{1} - \frac{n}{r} = 0 \rightarrow \quad r=n$$

$$\frac{1}{n!} \le \left(\frac{e}{n}\right)^n$$

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$$a(r) = \frac{r f'(r)}{f(r)}$$

$$b(r) = r a'(r) = r\left[\frac{f'}{f} + \frac{r f''}{f} - \frac{r(f')^2}{f^2}\right]$$

$$= r\left[\frac{f'}{f} + \frac{r f''}{f} - \right.$$

$$= \frac{r f'}{f} + \frac{r^2 f''}{f} - \frac{r^2 (f')^2}{f^2} \qquad 8.4.5$$

$$a(r) = \frac{re^z}{e^z} = r$$

$$a(r_n) = n \Rightarrow r_n = n$$

$$b(r) = ra'(r) = r.$$

$$\frac{1}{n!} \sim \frac{e^{r_n}}{r_n^n \sqrt{2\pi r_n}} = \frac{e^n}{n^n \sqrt{2\pi n}}$$

$$f(z) = e^{z+\frac{1}{2}z^2} \qquad f'(z) = e^{z+\frac{1}{2}z^2}(1+z)$$

$$a(r) \equiv r\frac{f'(r)}{f(r)} = \frac{r(1+r)\cancel{e^{r+\frac{1}{2}r^2}}}{\cancel{e^{r+\frac{1}{2}r^2}}} = r + r^2$$

$$b(r) = r\,a'(r) = r(1+2r) = r + 2r^2$$

let r_n sol to $a(r_n) = n$

$$\Rightarrow r_n + r_n^2 = n \qquad \Rightarrow r_n^2 + r_n - n = 0$$

$$r_n = \frac{-1 \pm \sqrt{1-4(-n)}}{2} = \frac{-1 \pm \sqrt{1+4n}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4n}$$

$$= -\frac{1}{2} \pm \sqrt{\frac{1}{4}+n} = \sqrt{n+\frac{1}{4}} - \frac{1}{2}$$

$$= \sqrt{n}\left(1+\frac{1}{4n}\right)^{1/2} - \frac{1}{2}$$

$$= \sqrt{n}\Big\{1 + \frac{1}{8n} - \frac{1}{}$$

$$e^{r_n + \frac{1}{2}r_n^2} = e^{r_n + \frac{1}{2}n - \frac{1}{2}r_n} = e^{\frac{1}{2}n + \frac{1}{2}r_n}$$

$$\left\{ r_n + r_n^2 = n \Rightarrow \frac{1}{2}r_n^2 = \frac{1}{2}n - \frac{1}{2}r_n \right\}$$

$$= e^{\frac{n}{2}} \cdot e^{\frac{r_n}{2}}$$

Then $e^{\frac{r_n}{2}}$

$$b(r_n) = r_n + 2r_n^2 = r_n + r_n^2 + r_n^2 = n + r_n^2$$

$$= n + \left\{ \sqrt{n} - \frac{1}{2} + \frac{1}{8\sqrt{n}} - O(n^{-3/2}) \right\}^2$$

$$= n + \left\{ n - \sqrt{n} + \frac{2}{8} + O(n^{-1}) + \frac{1}{4} + O(n^{-1/2}) + O(n^{-1}) \right\}$$

$$= 2n - \sqrt{n} + \frac{1}{4} \qquad n \gg 1$$

$$\sim 2n \qquad n \to \infty$$

$$r_n^n = \left\{ \sqrt{n} - \frac{1}{2} + \frac{1}{8\sqrt{n}} \cdots \right\}^n = n^{n/2} \left\{ 1 - \frac{1}{2\sqrt{n}} + \frac{1}{8n} + O(n^{-2}) \right\}^n$$

consider $\lim_{n \to \infty} \left\{ 1 - \frac{1}{2\sqrt{n}} \right\}^n$

type 1^∞ indeterminate

$$= \lim_{n\to\infty} \exp\left\{ n \ln\left\{1 - \frac{1}{2\sqrt{n}}\right\}\right\}$$

$$= \lim_{n\to\infty} \exp\left\{ \frac{\ln\left\{1-\frac{1}{2\sqrt{n}}\right\}}{\frac{1}{n}} \right\}$$

type $\frac{0}{0}$

$$= \lim_{n\to\infty} \exp\left\{ \frac{\frac{1}{1-\frac{1}{2\sqrt{n}}}\left(-\frac{1}{2}\left(-\frac{1}{2}\right)n^{-3/2}\right)}{\left(\frac{-1}{n^2}\right)} \right\}$$

$$= \lim_{n\to\infty} \exp\left\{ \frac{\frac{\frac{1}{4n^{3/2}}}{1-\frac{1}{2\sqrt{n}}}}{\frac{-1}{n^2}} \cdot \frac{\cancel{4}}{n^2} \right\}$$

$$\frac{\frac{1}{4n^{3/2}}}{1-\frac{1}{2\sqrt{n}}} \cdot (-n^2)$$

$$\frac{-\frac{n^{1/2}}{2}}{1-\frac{1}{2\sqrt{n}}} \quad \frac{2\sqrt{n}}{2\sqrt{n}}$$

$$= \frac{-n}{2\sqrt{n}-1} \qquad \text{type } \frac{\infty}{\infty}$$

$$= \lim_{n \to \infty} \exp \left\{ \frac{-1}{2(\frac{t}{2})n^{-1/2}} \right\}$$

$$= 1$$

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(1) $$\xi = \sum_s \binom{sL+1}{s} \frac{A^{-sL-1}}{sL+1} \Rightarrow \xi = \xi(A)$$

solutions $\xi^L \pm A\xi + 1 = 0$

$u = t\phi(u) \Rightarrow u = u(t)$

consider $\cancel{\xi = \frac{\xi^L + 1}{A}}$

$$\xi = \sum_s \binom{sL+1}{s} \frac{A^{-(sL+1)}}{sL+1}$$

$$\frac{d\xi}{dA} = \sum_s \binom{sL+1}{s} (-1) A^{-(sL+1)-1} = -\sum_s \binom{sL+1}{s} A^{-sL-2}$$

$$\xi = \sum_s \binom{sL+1}{s} \frac{(A^{-1})^{sL+1}}{sL+1} \Rightarrow \xi = \xi(A)$$

consider $\xi^L - A\xi + 1 = 0$

$$\xi = \frac{(1+\xi^L)}{A} = A^{-1}(1+\xi^L)$$

Then $\phi(\xi) = 1 + \xi^L$ & $t = A^{-1}$

By Lagrange-Bürmann Formula taking $F(u) = u$

$$[t^n]\{\xi(t)\} = \frac{1}{n}[\xi^{n-1}]\{1 \cdot (1+\xi^L)^n\}$$

$$= \frac{1}{n}[\xi^{n-1}]\left(\sum_{s=0}^{n} \binom{n}{s} \xi^{Ls}\right)$$

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Thus

$[t^n]\{\xi(t)\} =$

$s = ? \ni \quad sL = \cancel{n} \Rightarrow s = \frac{n}{L}$

$s = ? \ni \quad sL = n-1 \Rightarrow s = \frac{n-1}{L}$

$\therefore \ [t^n]\,\xi(t) = \frac{1}{n}\binom{n}{\frac{n-1}{L}}\xi^{n-1}$ What is wrong?

(2) Darboux's method: $v(z)$ analytic in disk $|z| < 1+\eta$

$$v(z) = \sum v_j (1-z)^j$$

$$[z^n]\{(1-z)^\beta v(z)\} = [z^n]\left\{\sum_{j=0}^{m} v_j (1-z)^{\beta+j}\right\} + O(n^{-m-\beta-2})$$

$$\tau^2 - 2x\tau + 1 = 0$$

$$\tau = \frac{2x \pm \sqrt{4x^2-4}}{2} = \frac{x \pm \sqrt{x^2-1}}{\cancel{2}} = \frac{x \pm i\sqrt{1-x^2}}{\cancel{2}}$$

How do?

(3) $u = t\phi(u)$

By the Lagrange Inversion Formula:

$$[u^n]\left\{\frac{u}{n}F'(u)\,\phi(u)^n\right\}$$

$$= [t^n]\{F(u(t))\}$$

$$[u^3]\{A + Bu + Cu^2 + Du^3 + Eu^4\}$$

$$= D$$

$$[u^{3-1}]\{\underline{\qquad\qquad}\}$$

$$[u^3]\{u\cdot$$

$\Rightarrow$ Pick $F(u) = n\ln(u)$

then $F'(u) = \dfrac{n}{u}$

+ Above becomes

$$[u^n]\{\phi(u)^n\} = [t^n]\{n\ln(u(t))\}$$

— — —

If

$$[u^n]\{\phi(u)\}^n = [t^n]\left\{\frac{tu'(t)}{u(t)}\right\}$$

$$= [t^n]\left\{\frac{t}{u(t)}\cdot\frac{\phi(u(t))}{(1-t\phi'(u(t)))}\right\}$$

$$u = t\phi(u)$$

$$u' = \phi + t\phi' u' \quad\rightarrow\quad u' = \frac{\phi}{1-t\phi'}$$

Then

$$[u^n]\{\phi^n(u)\} = [t^n]\left\{ \frac{1}{1-t\phi'(u(t))} \right\}$$

(4) $V_n = [x^n](1+x+x^2)^n$

(a) ~~Bx~~ By ex 3

$$[x^n]\{(1+x+x^2)^n\} = [t^n]\left\{\cancel{\frac{tv'(t)}{v(t)}}\right\}$$

consider $\phi(x) = 1+x+x^2$ & $x = t\phi(x) = t(1+x+x^2)$

$$= [t^n]\left\{\frac{tx'(t)}{x(t)}\right\}$$

But $\frac{d}{dt}$ of $x = t\phi(x)$ is

$$x'(t) = 1+x+x^2 + t(x' + 2xx')$$

$$= \frac{x}{t} + t(1+2x)x'$$

$$(1-t(1+2x))x' = \frac{x}{t} \Rightarrow \frac{tx'}{x} = \frac{1}{(1-t(1+2x))}$$

Since x is a solution to $x = x(t)$

$$x = t(1+x+x^2)$$

$$x^2+x+1-\frac{x}{t} = 0$$

$$x^2+(1-\tfrac{1}{t})x+1 = 0 \qquad x = \frac{-[1-\frac{1}{t}] \pm \sqrt{(1-\frac{1}{t})^2-4}}{2}$$

$$\cancel{1+}2x = -1+\frac{1}{t} \pm \sqrt{(1-\tfrac{1}{t})^2-4}$$

$$2x+1 = \frac{1}{t} \pm \sqrt{(\ \ )^2-4}$$

$$t(2x+1) = 1 \pm t\sqrt{(\quad)^2 - 4}$$

$$-t(2x+1) = -1 \mp t\sqrt{(\quad)^2 - 4}$$

$$1 - t(2x+1) = \pm t\sqrt{(\quad)^2 - 4}$$

$$\therefore \quad \frac{tx'}{x} = \pm \frac{1}{t\sqrt{(1-\frac{1}{t})^2 - 4}} = \pm \frac{1}{\sqrt{t^2(1-\frac{1}{t})^2 - 4t^2}}$$

$$= \frac{\pm 1}{\sqrt{(t-1)^2 - 4t^2}} = \frac{\pm 1}{\sqrt{t^2 - 2t + 1 - 4t^2}}$$

$$= \frac{\pm 1}{\sqrt{1 - 2t - 3t^2}}$$

$$\therefore \quad [x^n]\{(1+x+x^2)^n\} = [t^n]\left\{\frac{\pm 1}{\sqrt{1-2t-3t^2}}\right\}$$

How know only use + sign?

since

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n \geq 0} P_n(x) t^n$$

$$\frac{1}{\sqrt{1-2t-3t^2}} = \frac{1}{\sqrt{1-2t+(\sqrt{3}ti)^2}}$$

$$= \frac{1}{\sqrt{1 - 2\left(\frac{1}{i\sqrt{3}}\right)(\sqrt{3}ti) - (\sqrt{3}ti)^2}}$$

$$\equiv \sum_{n\geq 0} P_n\left(\tfrac{1}{i\sqrt{3}}\right)(i\sqrt{3}t)^n = \sum_{n\geq 0} i^n 3^{n/2} P_n\left(\tfrac{1}{i\sqrt{3}}\right) t^n$$

so

$$[x^n]\{(1+x+x^2)^n\} = i^n 3^{n/2} P_n\left(\tfrac{-i}{\sqrt{3}}\right)$$

P_n even ?

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(5) $$S_p(n) = \sum_{k=0}^{n} \binom{pn}{k}$$

$$S(x) \equiv \sum_{n \geq 0} S_p(n) x^n = \cancel{\sum_{k=0}^{n} \sum_{n \geq 0}} = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{pn}{k} \right) x^n$$

$$=$$

$$\sum_{k=0}^{pn} \binom{pn}{k} = 2^{pn}$$

$$\|$$

$$\sum_{k=0}^{n} \binom{pn}{k} + \sum_{k=n+1}^{2n} \binom{pn}{k} + \cdots + \sum_{k=(p-1)n+1}^{pn} \binom{pn}{k} =$$

(a) $$S_p(n) = [x^n] \left\{ \frac{(1+x)^{pn}}{1-x} \right\}$$

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(7)

$(1+t_n)^n$

(a) $t_n = n^a$ $\quad 0<a<1$ $\quad -1<-a<0$

$0<1-a<1$

$\lim_{n\to\infty} (1+n^a)^n$ $\quad$ ~~type~~

$= n^{an} \cdot \lim_{n\to\infty} (1+n^{-a})^n$ $\quad$ type 1^∞ indet.

$= n^{an} \exp\left[n \ln(1+n^{-a})\right]$ $\quad$ ~~$\ln(1+x) = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^p}{p}$~~

$\cong n^{an} \exp\left[n\left[1-n^{-a}+\frac{1}{2}n^{-2a} - \frac{1}{3}n^{-3a}+\frac{1}{4}n^{-4a}+O(n^{-5a})\right]\right]$ $\quad$ $\ln(1+x) = 1-x+\frac{x^2}{2}-\frac{x^3}{3}\cdots$

$\cong n^{an} \exp\left[n - n^{1-a} + \frac{1}{2}n^{1-2a} - \frac{1}{3}n^{1-3a} + O(n^{1-4a})\right]$

continue terms untill they become $o(1)$ i.e. exponent is negative!!

$\cong n^{an}\left[\exp\left[1-n^{-a}+\frac{1}{2}n^{-2a}-\frac{1}{3}n^{-3a}+O(n^{-4a})\right]\right]^n$

$= n^{an}\left[\exp(1)\exp\left(-n^{-a}+\frac{1}{2}n^{-2a}-\frac{1}{3}n^{-3a}+O(n^{-4a})\right)\right]^n$

$$= n^{an} e^{n} \cdot \exp\left[n\left(-n^{-a} + \tfrac{1}{2} n^{-2a}\right.\right.$$

lost 1

$$= n^{an} e^{n} \left[\exp\left[-n^{1-a}\left(1 - \tfrac{1}{2} n^{-a} + \tfrac{1}{3} n^{-2a} + O(n^{-3a})\right)\right]\right]^{n}$$

$$= n^{an} e^{n} \exp\left[-n^{1-a}\left(1 - \tfrac{1}{2} n^{-a} + O(n^{-2a})\right)\right]$$

$$\cong n^{an} e^{n} e^{-n^{1-a}} \qquad n \to +\infty.$$

(b) $(1+t_n)^n = (1+n^{-a})^n$

$$= \exp\left[n \ln(1+n^{-a})\right]$$

$$= \exp\left[n\left(1 - n^{-a} + \tfrac{1}{2} n^{-2a} - \tfrac{1}{3} n^{-3a} + \tfrac{1}{4} n^{-4a} + O(n^{-5a})\right)\right]$$

$$= \cancel{e}\, e^{n} \exp\left[-n^{1-a}\left(1 - \tfrac{1}{2}\cancel{a} n^{-a} + \tfrac{1}{3} n^{-2a} + O(n^{-3a})\right)\right]$$

$$\cong \cancel{e}\,e\; e^{n} e^{-n^{1-a}} \qquad n \to +\infty$$

(c) $\left(1+n^{-a}\ln(n)\right)^n$

$$= \exp\left[n\log\left(1+n^{-a}\ln(n)\right)\right]$$

$\lim\limits_{n\to\infty} \frac{\ln(n)}{n^a}$ $\frac{\infty}{\infty}$

$= \frac{1/n}{an^{a-1}} \to 0.$

$$= \exp\left[n\left(1-n^{-a}\ln(n) + \tfrac{1}{2}n^{-2a}(\ln(n))^2 + O(n^{-3a}(\ln n)^3)\right)\right]$$

$$= e^n \exp\left[-n^{1-a}\ln(n) + \cdots\right]$$

$-1 < a-2 < 0$

$$= e^n.$$

$1 < a < 2$

$-2 < -a < -1$

$-1 < 1-a < 0$

$0\cdot\infty$

$\frac{\ln(n)}{n^{a-1}} \to \frac{1/n}{(a-1)n^{a-2}} \to 0$