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(2.2) Explicit Runge-Kutta Formulas have the following form

$$y_{n+1} = y_n + h\sum_{j=1}^{s} A_j f(t_{n,j}, y_{n,j})$$

w/ $y_{n,j}$ given explicitly in terms of previous $y_{n,j}$'s i.e. $y_{n,j} = y_n + h\sum_{k=1}^{j-1} \beta_{j,k} f_{n,k}$.

w/ $f_{n,j} = f(t_n + \alpha_j h_n, y_{n,j})$

Using the mid point rule to evaluate the explicit Runge-Kutta expression gives

$$y_{n+1} = y_n + h f\left(t_n + \frac{h}{2}, y_{n,1/2}\right)$$

w/ $y_{n,1/2} \cong y\left(t_n + \frac{h}{2}\right)$ so using Eulers method to evaluate $y_{n,1/2}$ gives $y_{n,1/2} = y_n + \frac{h}{2} f(t_n, y_n)$, thus the entire update step is

$$y_{n,1/2} = y_n + \frac{h}{2} f(t_n, y_n)$$

$$y_{n+1} = y_n + h f\left(t_n + \frac{h}{2}, y_{n,1/2}\right).$$

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Ex 2.3 The equations of condition for a Runge-Kutta code of order p are given on pg 81 of the book. they are

$$\frac{1}{k} = \sum_{j=1}^{s} \gamma_j \alpha_j^{k-1} \quad \text{for} \quad k = 1, 2, \dots, p$$

Then for the given method to be of 2nd order we require

$$\frac{\cancel{k}}{1}$$

$$Y_{n,1} = Y_n \quad f_{n,1} = f(t_n, Y_{n,1})$$

$$Y_{n,j} = Y_n + h_n \sum_{k=1}^{j-1} \beta_{j,k} f_{n,k} \quad j = 1, 2, \dots, s$$

$$f_{n,j} = f(t_n + \alpha_j h_n, Y_{n,j}) \quad j = 1, 2, \dots, s$$

$$y_{n+1} = y_n + h_n \sum_{j=1}^{s} \gamma_j f_{n,j}$$

hence

$$\frac{1}{1} = \sum_{j=1}^{2} \gamma_j \alpha_j^0 = \sum_{j=1}^{2} \gamma_j \quad \checkmark$$

$$\text{\&} \quad \frac{1}{2} = \sum_{j=1}^{2} \gamma_j \alpha_j^1 = \gamma_1 \cdot 0 + \gamma_2 \alpha_1$$

$$= \quad \frac{1}{2} = \gamma_2 \alpha_1$$

$$y_{n+1} = y_n + h\left[r_1 f_{n,1} + r_2 f_{n,2} \right]$$

$$f_{n,1} = f(t_n, y_n)$$

$$f_{n,2} = f(t_n + \alpha_1 h, y_n + h\beta_{1,0} f_{n,1})$$

so $$y_{n+1} = y_n + h\left[r_1 f(t_n, y_n) + r_2 f(t_n + \alpha_1 h, y_n + h\beta_{1,0} f(t_n, y_n)) \right]$$

w/ $f(t, u)$ scalar

$$u(t_{n+1}) \cong y_{n+1}$$

$$u(t_{n+1}) = \underset{\displaystyle y_n}{\underset{|||}{u(t_n)}} + \underset{\displaystyle hf(t_n, y_n)}{\underset{|||}{hu'(t_n)}} + \frac{h^2}{2} u''(t_n) + \frac{h^3}{6} u'''(t_n) + O(h^4)$$

$$= y_n + hf(t_n, y_n) + \frac{h^2}{2} u''(t_n) + \frac{h^3}{6} u'''(t_n) + O(h^4)$$

so

$$+ \quad y_{n+1} = y_n + hr_1 f(t_n, y_n) + hr_2 f(t_n + \alpha_1 h, y_n + h\beta_{1,0} \underline{\underline{f(t_n, y_n)}})$$

$$= y_n + hr_1 f(t_n, y_n) + hr_2 \left[\cancel{f(t_n, y_n)} + \right.$$

$$\left[f(t_n, y_n + h\beta_{1,0} f(t_n, y_n) \right.$$

$$= h\gamma_2 \Big[f(t_n,y_n) + \alpha_1 h f_t(t_n,y_n) + h\beta_{1,0} f_y(t_n,y_n) f(t_n,y_n) + O(h^2) \Big]$$

$\therefore$

$$y_{n+1} = y_n + [h\gamma_1 + h\gamma_2] f(t_n,y_n) + h^2\gamma_2\alpha_1 f_t(t_n,y_n) + h^2\gamma_2\beta_{1,0} f_y(t_n,y_n) f(t_n,y_n) + O(h^3)$$

So

$$u(t_{n+1}) - y_{n+1} = h[1 - \gamma_1 - \gamma_2] f(t_n,y_n) + \frac{h^2}{2} u''(t_n) - h^2\gamma_2\alpha_1 f_t(t_n,y_n) - h^2\gamma_2\beta_{1,0} f_y(t_n,y_n) f(t_n,y_n) + O(h^3)$$

But $u''(t_n) = ?$

$u'(t) = f(t,u)$

So $u''(t) = f_t + f_u u' = f_t + f_u \cdot f$

So $u''(t_n) = f_t(t_n,y_n) + f_u(t_n,y_n) f(t_n,y_n)$

$$\therefore \quad u(t_{n+1}) - y_{n+1} = h\left[1 - \gamma_1 - \gamma_2\right] f(t_n, y_n)$$

$$+ h^2 \Big[\tfrac{1}{2} f_t(t_n, y_n) + \tfrac{1}{2} f_u(t_n, y_n) \cdot f(\cdot,\cdot)$$

$$- \gamma_2 \alpha_1 f_t(\cdot,\cdot) - \gamma_2 \beta_{1,0} f_u(\cdot,\cdot) f(\cdot,\cdot) \Big] + O(h^3)$$

$\Rightarrow$ Eq's of consistency become:

$$\left.\begin{aligned} 1 &= \gamma_1 + \gamma_2 \\ \tfrac{1}{2} &= \gamma_2 \alpha_1 \\ \tfrac{1}{2} &= \gamma_2 \beta_{1,0} \end{aligned}\right] * .$$

(2.4)

$$Y_{n+1} = Y_n + h f_{n,2}$$

$$Y^*_{n+1} = Y_n + \frac{h}{9}\left[2f_{n,1} + 3f_{n,2} + 4f_{n,3}\right]$$

$f_{n,1} = f(t_n, Y_n)$

$$\begin{array}{c|c} \alpha & \beta \\ \hline & \gamma \end{array}$$

α vector/array.

β matrix

γ vector array

Euler - Heun ~~R-K~~ R-K method:

$$Y_{n,1} = Y_n + h f(t_n, Y_n)$$

$$Y_{n+1} = Y_n + h\left[\frac{1}{2} f(t_n, Y_n) + \frac{1}{2} f(t_{n+1}, Y_{n,1})\right]$$

$$\Rightarrow \quad \bar{\alpha} = (0, 1)$$

$$\bar{\gamma} = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$[\beta] = \begin{matrix} 0 \\ 1 \end{matrix}$$

$$Y_{n,1} = Y_n \qquad f_{n,1} = f(t_n, Y_{n,1})$$

$j = 2$

$j = 3$

$\vdots$

$j = s$

$$Y_{n,j} = Y_n + h_n \sum_{k=1}^{j-1} \beta_{jk} f_{n,k} \qquad f_{n,j} = f(t_n + \alpha_j h_n, Y_{n,j})$$

Finally

$$Y_{n+1} = Y_n + h_n \sum_{j=1}^{s} \gamma_j f_{n,j}$$

Euler - Heun:

$$y_{n,1} = y_n \qquad f_{n,1} = f(t_n, y_{n,1})$$

$j = 2$ $$y_{n,2} = y_n + h\beta_{2,1} f_{n,1} \qquad f_{n,2} = f(t_n + \alpha_2 h, y_{n,2})$$

+ Finally: $$y_{n+1} = y_n + h \sum_{j=1}^{2} \gamma_j f_{n,j}$$

$$y_{n,1} = y_n \qquad f_{n,1} = f_{n,1}(t_n, y_{n,1})$$

$$y_{n,2} = y_n + h f_{n,1}(t_n, y_{n,1}) \qquad f_{n,2} = f(t_n + h, y_{n,2})$$

$$y_{n+1} = y_n + h\left[\frac{1}{2} f_{n,1} + \frac{1}{2} f_{n,2} \right]$$

$$\begin{array}{c|c} \alpha & \beta \\ \hline & \gamma \end{array}$$

typo on Table 2.2

$$\therefore \quad \begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$Y_{n,1} = Y_n \qquad f_{n,1} = f(t_n, Y_{n,1})$$

$$Y_{n,2} = Y_n + h\,\tfrac{1}{2} f_{n,1} \qquad f_{n,2} = f\left(t_n + \tfrac{h}{2}, Y_{n,2}\right)$$

+ ~~Finally $Y_{n+1} = Y_n + 0 \cdot f_{n,1} + h f_{n,2}$~~

$$Y_{n,3} = Y_n + \tfrac{3}{4} h f_{n,2} \qquad f_{n,3} = f\left(t_n + \tfrac{3}{4} h, Y_{n,3}\right)$$

+ Finally $Y_{n+1} = Y_n + h\left[\tfrac{2}{9} f_{n,1} + \tfrac{3}{9} f_{n,2} + \tfrac{4}{9} f_{n,3}\right]$

So $\alpha = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix}$; $\gamma = \begin{pmatrix} \frac{2}{9} \\ \frac{3}{9} \\ \frac{4}{9} \end{pmatrix}$; $\beta = \begin{pmatrix} 0 & & \\ 1/2 & & \\ 0 & 3/4 & \end{pmatrix}$

$\therefore$ Butcher tableau is:

$$\begin{array}{l|c|ccc} & \alpha & & \beta & \\ j=1 & 0 & \cdot & & \\ j=2 & \frac{1}{2} & \frac{1}{2} & & \\ j=3 & \frac{3}{4} & 0 & 3/4 & \\ \hline & & \frac{2}{9} & \frac{3}{9} & \frac{4}{9} \\ & & & \gamma & \end{array}$$

$$|est| \le \tau_r |y^*_{n+1}| + \tau_a \qquad h_{new} = 6h$$

$$est = h\left[\frac{2}{9} f_{n,1} + \underbrace{\left(\frac{1}{3} - 1\right)}_{-\frac{2}{3}} f_{n,2} + \frac{4}{9} f_{n,3}\right]$$

$$= \frac{h}{9}\left[2 f_{n,1} - 6 f_{n,2} + 4 f_{n,3}\right] = \frac{2h}{9}\left[f_{n,1} - 3 f_{n,2} + 2 f_{n,3}\right]$$

Don't follow how to do the rest of this problem

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$$y_{n+1} = h\gamma f(t_{n+1}, y_{n+1}) + \psi$$

$$\Rightarrow \; y_{n+1}^{[m+1]} = \psi + h\gamma\left[f(t_{n+1}, y_{n+1}^{[m]}) + J\left(y_{n+1}^{[m+1]} - y_{n+1}^{[m]}\right)\right]$$

let $y_{n+1}^{[m+1]} = y_{n+1}^{(m)} + \Delta_m$ then the above becomes

$$y_{n+1}^{[m]} + \Delta_m = \psi + h\gamma\left[f(t_{n+1}, y_{n+1}^{(m)}) + J\,\Delta_m\right]$$

$$\Rightarrow (I - h\gamma J)\Delta_m = \psi + h\gamma f(t_{n+1}, y_{n+1}^{[m]}) - y_{n+1}^{[m]} \qquad \text{eq } 2.36 \checkmark$$

w/ $J \equiv \frac{\partial f}{\partial y}(t_{n+1}, y_{n+1}^{[m]})$

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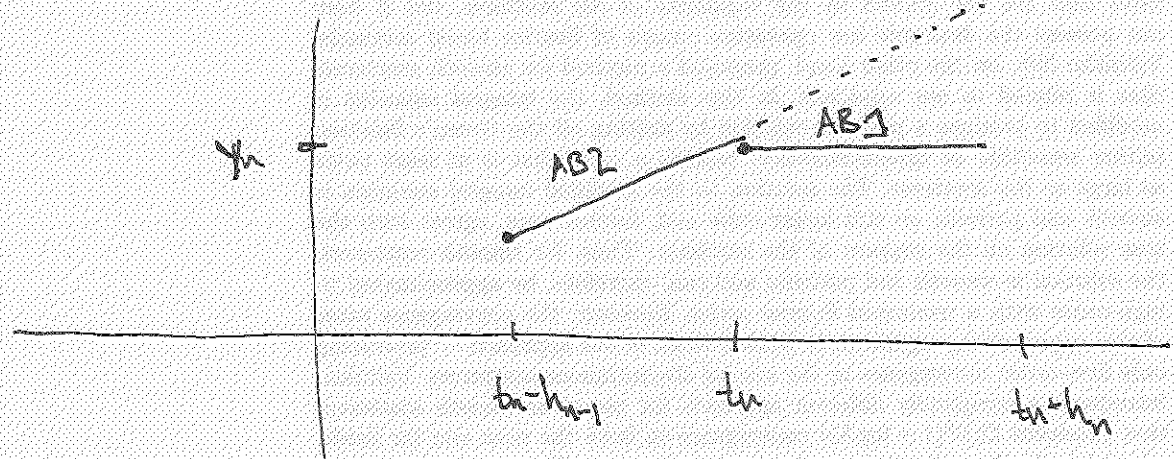
25 AB2:

$$\int_{t_n}^{t_n+h} f(x,u(x))\,dx$$

take $t_{n,j} = t_{n-j}$ $j \geq 1$ Adams Bashforth

$u' = f(t,u)$ $u(t_n) = y_n$

$$u(t_n+h_n) = y_n + \int_{t_n}^{t_n+h_n} f(x,u(x))\,dx \qquad f(x,u) = \cancel{\ }$$



AB1: interpolate f: ~~between~~ at t_n, ~~t_n+h_n~~ ~~w/ a~~ ~~constant~~.

AB2: " f: at $t_n + t_{n-1} \equiv f(t_n,y_n), f(t_{n-1},y_{n-1})$

$$\underset{f_n}{|||} \qquad \underset{f_{n-1}}{|||}$$

~~$f(x,u)$~~

~~$f(x,u)$~~ $$f(t,u) \cong \frac{(t-t_{n-1})}{(t_n-t_{n-1})} f_n + \frac{(t-t_n)}{(t_{n-1}-t_n)} f_{n-1}$$ Lagrange interpolating polynomial

So AB2 is

$$\cancel{y_{n+1} =} \quad \text{or} \quad y_{n+1} = y_n + \int_{t_n}^{t_n+h_n} \left(\frac{(t-t_{n-1})}{(t_n - t_{n-1})} f_n + \frac{(t-t_n)}{(t_{n-1}-t_n)} f_{n-1} \right) dt$$

$$= y_n + \frac{f_n}{h_{n-1}} \frac{(t-t_{n-1})^2}{2} \Bigg|_{t_n}^{t_n+h_n} + \frac{f_{n-1}}{2(-h_{n-1})} (t-t_n)^2 \Bigg|_{t_n}^{t_n+h_n}$$

$$= y_n + \frac{f_n}{2h_{n-1}} (h_{n-1}+h_n)^2 - \frac{f_n}{2h_{n-1}} h_{n-1}^2 - \frac{f_{n-1}}{2h_{n-1}} h_n^2 + \frac{f_{n-1}}{2h_{n-1}} \cdot 0$$

$$= y_n + \frac{f_n}{2h_{n-1}} \left(\cancel{h_{n-1}^2} + 2h_{n-1}h_n + h_n^2 - \cancel{h_{n-1}^2} \right) - \frac{f_{n-1} h_n^2}{2h_{n-1}}$$

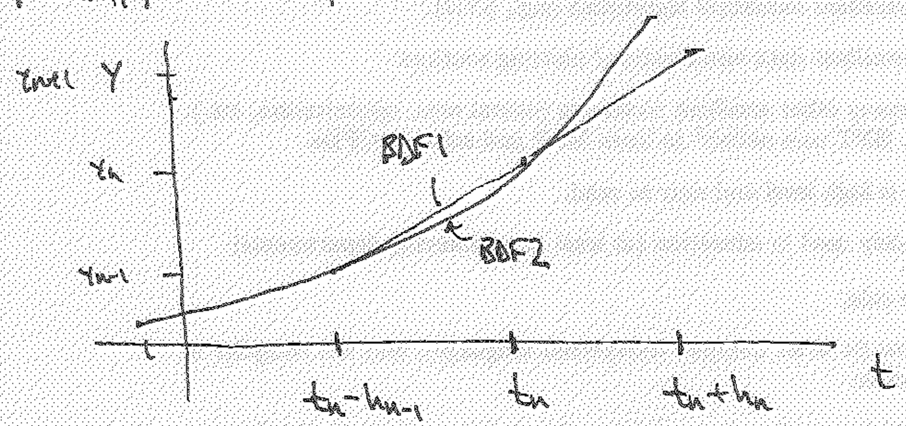
$$= y_n + h_n \left[\left(1 + \frac{1}{2}\left(\frac{h_n}{h_{n-1}}\right) \right) f_n - \left(\frac{1}{2}\left(\frac{h_n}{h_{n-1}}\right) \right) f_{n-1} \right] \quad \checkmark$$

define $r \equiv \frac{h_n}{h_{n-1}}$ then

$$= y_n +$$

For BDF2 (Rather than approximating f at previous mesh points we approximate $y(t)$ & at y_{n-j} $j \geq 0$ w/ req that the polynomial satisfy the ODE at $t = t_{n+1}$ i.e. it collocates the ODE at t_{n+1} or

$$P'(t_{n+1}) = f(t_{n+1}, P(t_{n+1})) = f(t_{n+1}, y_{n+1})$$



$$y_{n+1} = y_n + h_n \int_{t_n}^{t_n+h_n} f(t', u(t'))\,dt' \cong y_n + h_n$$

$$y_{n+1} = P(t_{n+1}; \{y_{n-j}\})$$

$$+ \quad P'(t_{n+1}; \{y_{n-j}\}) = f(t_{n+1}, y_{n+1})$$

so 1st order extrapolation of $y_{n+1} \cong \left.\underbrace{\frac{(t-t_n)}{(t_{n-1}-t_n)} y_{n-1} + \frac{(t-t_{n-1})}{(t_n - t_{n-1})} y_n}_{\equiv P(t; \{y_n, y_{n-1}\})}\right|_{t=t_{n+1}}$

+ collocation requirement $\quad P'_{BDF1}(t_{n+1}; \{..\}) = f(t_{n+1}, y_{n+1})$

$$P_{BDF1}(t; \{y_n, y_{n-1}\}) = -\frac{y_{n-1}}{h_{n-1}}(t - t_n) + \frac{y_n}{h_n}(t - t_{n-1})$$

$$P'_{BDF1}(t; \{..\}) = -\frac{y_{n-1}}{h_{n-1}} + \frac{y_n}{h_n} \overset{\text{set}}{=}$$

$$P'(t_{n+1}) = f(t_{n+1}, P(t_{n+1}))$$

$$P_{BDF1}(t_{n+1}, \{\cancel{y_{n-1}}, y_n\}) = y_{n+1} + P'_{BDF1}(t_{n+1}, \{..\}) = f(t_{n+1}, y_{n+1})$$

$$P_{BDF1}(t; \{y_{n+1}, y_n\}) = \frac{(t - t_{n+1})}{(t_n - t_{n+1})} y_n + \frac{(t - t_n)}{(t_{n+1} - t_n)} y_{n+1}$$

$$P'_{BDF1}(t; \{..\}) = \frac{y_n}{-h_n} + \frac{y_{n+1}}{h_n} = \frac{y_{n+1} - y_n}{h_n} \equiv f(t_{n+1}, y_{n+1})$$

For BDF2

$$P_{BDF2}(t) = y_{n+1}\frac{(t - t_n)(t - t_{n-1})}{(t_{n+1} - t_n)(t_{n+1} - t_{n-1})} + y_n\frac{(t - t_{n+1})(t - t_{n-1})}{(t_n - t_{n+1})(t_n - t_{n-1})} + y_{n-1}\frac{(t - t_{n+1})(t - t_n)}{(t_{n-1} - t_{n+1})(t_{n-1} - t_n)}$$

$$P_{BDF2}(t) = \frac{y_{n+1}(t-t_n)(t-t_{n-1})}{h_n(h_n+h_{n-1})} + \frac{y_n(t-t_{n+1})(t-t_{n-1})}{(-h_n)(h_{n-1})}$$

$$\underbrace{t_{n+1}-t_n}_{h_n} + \underbrace{t_n - t_{n-1}}_{h_{n-1}}$$

$$+ \frac{y_{n-1}(t-t_{n+1})(t-t_n)}{(-1)(h_n+h_{n-1})(-1)(h_{n-1})}$$

$$= A(t-t_n)(t-t_{n-1}) + B(t-t_{n+1})(t-t_{n-1}) + C(t-t_{n+1})(t-t_n)$$

$$P_{BDF2}'(t) = \frac{\cancel{y_{n+1}}}{\cancel{h_n}}$$

$$= A(t-t_{n-1}) + A(t-t_n) + B(t-t_{n-1}) + B(t-t_{n+1})$$

$$+ C(t-t_n) + C(t-t_{n+1})$$

$$P_{BDF2}'(t_{n+1}) = A(h_n+h_{n-1}) + Ah_n + B(h_n+h_{n-1}) + \cancel{B(\ldots)}$$

$$+ Ch_n$$

$$= \frac{y_{n+1}}{h_n} + \frac{y_{n+1}}{(h_n+h_{n-1})} + \frac{y_n(h_n+h_{n-1})}{(-h_n)(h_{n-1})} + \frac{y_{n-1}(h_n)}{(h_n+h_{n-1})(h_{n-1})}$$

so $p'(t_{n+1}; \{y_{n+1}, y_n, y_{n-1}\}) = f(t_{n+1}, y_{n+1})$

$$\parallel$$

$$y_{n+1}\left(1 + \frac{h_n}{h_n + h_{n-1}}\right) - \frac{(h_n + h_{n-1})}{h_{n-1}} y_n + \frac{h_n^2}{h_{n-1}(h_n + h_{n-1})} y_{n-1} = h_n f(t_{n+1}, y_{n+1})$$

$$\Rightarrow y_{n+1}\left(\frac{2h_n + h_{n-1}}{h_n + h_{n-1}}\right) - \left(1 + \frac{h_n}{h_{n-1}}\right) y_n + \left(\frac{(h_n/h_{n-1})^2}{(1 + h_n/h_{n-1})}\right) y_{n-1} = \text{''}$$

$$\Rightarrow \left(\frac{2+r}{1+r}\right) y_{n+1} - (1+r) y_n + \left(\frac{r^2}{1+r}\right) y_{n-1} = \text{''}$$

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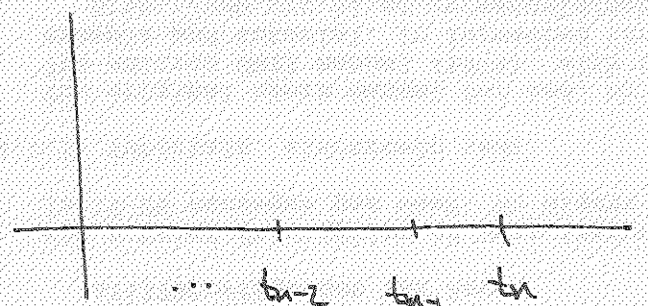
(2b) AM2 is the following scheme:



$f_{n-j} = f(t_{n-j}, y_{n-j})$

taking $t_{n-j} = t_{n-j}$ are usually called Adam's-Bashforth formula's.

Adam-Moulton: we interpolate f_{n+1}

AM2 is the trapezoidal rule

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

$$lte_n \equiv y_{n+1} - \left(y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)\right) \qquad t_{n+1} \equiv t_n + h$$

$$= y_n + h y'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{6} y'''(t_n) + O(h^4)$$

$$- y_n - \frac{h}{2}\left(f(t_n, y_n)\right) - \frac{h}{2}\Big[\qquad +$$

$$f\left(t_n + h, y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + O(h^4)\right)$$

w/ $y' = f$

& $y'' = f_t + f_y y'$

& $f(t+h, y + h(y' + \frac{h}{2}y'' + \frac{h^2}{6}y''' + O(h^3)))$

$= f(t,y) + hf_t + h(y' + \frac{h}{2}y'' + O(h^2))f_y$

$+ \frac{h^2}{2}f_{tt} + h\,h(y' + \frac{h}{2}y'' + O(h^2))f_{ty} + \frac{h^2}{2}(y' + \frac{h}{2}y'' + O(h^2))^2 f_{yy}$

$+ O(h^3)$

$= f + hf_t + hy'f_y + \frac{h^2 y''}{2}f_y + \frac{h^2}{2}f_{tt} + h^2 y' f_{ty} + O(h^3)$

$+ \frac{h^2}{2}y'^2 f_{yy} + O(h^3)$

Then

$$\text{lten} = h\cancel{f} + \frac{h^2}{2}(\cancel{f_t} + \cancel{f_y y'}) + \frac{h^3}{6}(\underbrace{f_{tt} + 2f_{yt}y' + f_{yy}y'^2 + y''f_y}_{y'''(t_n)})$$

$$- \frac{h}{2}\cancel{f} - \frac{h}{2}\left[\cancel{f} + \cancel{hf_t} + \cancel{hy'f_y} + \frac{h^2 y''}{2}f_y + \frac{h^2}{2}f_{tt} + h^2 y' f_{ty} + \frac{h^2}{2}y'^2 f_{yy}\right]$$

$+ O(h^4)$

since $\quad y''' = f_{tt} + f_{ty}y' + f_{ty}y' + f_{yy}y'^2 + f_y y''$

$$= f_{tt} + 2f_{ty}y' + f_{yy}y'^2 + f_y y''$$

we have

$$\text{lten} = \frac{h^3}{6} y'''(t_n) - \frac{h^3}{4}\Big[\underbrace{y'' f_y + f_{tt} + 2y' f_{ty} + y'^2 f_{yy}}_{y'''(t_n)}\Big] + O(h^4)$$

$$\therefore \left(\frac{1}{6} - \frac{1}{4}\right) h^3 y'''(t_n) \qquad \frac{4-6}{24} = \frac{-1}{12}$$

$$= -\frac{1}{12} h^3 y'''(t_n) + O(h^4)$$

Ex 27

$$y' = \cancel{} - y$$

AM2 ⇒ trapezoidal rule

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

$$= y_n + \frac{h}{2}(-y_n - y_{n+1})$$

$$= y_n - \frac{h}{2}(y_n + y_{n+1}) = \left(1 - \frac{h}{2}\right)y_n - \frac{h}{2}y_{n+1}$$

$$y_{n+1} = \frac{\left(1 - \frac{h}{2}\right)y_n}{\left(1 + \frac{h}{2}\right)}$$

From Ex 26 we have $\text{lten} = -\frac{h^3}{12}y^{(3)}(t_n) + O(h^4)$

~~Substitute analytical stuff~~

$$y' = -y \quad \text{so} \quad y'' = -y' = +y \;\; \& \;\; y''' = y' = -y$$

$$\therefore \quad \text{lten} \equiv y(t_n + h) - y_{n+1} = \frac{1}{12}h^3 y(t_n) + O(h^4)$$

- PECE stands for predict-evaluate-f-correct-evaluate f.

Heun's method:

$$y_{n+1} = y_n + hf(t_n, y_n) \qquad \text{AB1 : interpolate } y_{n+1} \text{ from } y_n$$

$$y_{n+1} = y_n + h\left[\tfrac{1}{2}f(t_n, y_n) + \tfrac{1}{2}f(t_{n+1}, y_{n+1})\right] \qquad \text{AM2 :}$$

Then

$$lte_n = y_{n+1} - y_n - \frac{h}{2}\left[f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))\right]$$

$$= y_{n+1} - y_n - \frac{h}{2}\Big[f + f + hf_t + hf\cdot f_y + \frac{h^2}{2}f_{tt} + h^2 f\cdot f_{ty}$$

$$+ h^2 f^2 f_{yy} + O(h^3)\Big]$$

$$= \cancel{y_{n+1}} - y_n$$

$$= \cancel{\quad}$$

$$= y_n + h\cancel{y'} + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + O(h^4) - y_n$$

$$-\frac{h}{2}\left[2\cancel{f} + hf_t + hff_y + \cancel{\quad} + \frac{h^2}{2}\left(f_{tt} + 2ff_{ty} + 2f^2 f_{yy}\right)\right] + O(h^4)$$

w/ $y' = -y \;;\; y'' = +y \;;\; y''' = -y$

$f_t = 0$

$$\therefore \quad lte_n = \cancel{\frac{h^2}{2}} y - \frac{h^3}{6} y$$

$$- \frac{h}{2} \left[+ h \cancel{y} \cdot 1 + \frac{h^2}{2} \left(2y^2 \cdot 0 \right) \right] + O(h^4)$$

$$= \frac{-h^3}{6} y \qquad \checkmark$$

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(28)

eq 229 is $y_{n+1} + \frac{3}{2}y_n - 3y_{n-1} + \frac{1}{2}y_{n-2} - 3hf(t_n, y_n) = 0$

Expanding y_{n+1} in a taylor series about t_n gives

$$y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \frac{h^3}{6}y'''(t_n) + O(h^4)$$

$$+ \frac{3}{2}y_n$$

$$-3\left[y_n - hy'(t_n) + \frac{h^2}{2}y''(t_n) - \frac{h^3}{6}y'''(t_n) + O(h^4)\right]$$

$$+ \frac{1}{2}\left[y_n - 2hy'(t_n) + \frac{4h^2}{2}y''(t_n) - \frac{8h^3}{6}y'''(t_n) + O(h^4)\right]$$

$$- 3hf(t_n, y_n) = 0$$

$$\Rightarrow \left(1 + \frac{3}{2} - 3 + \frac{1}{2}\right)y_n + h\left(1 + 3 - 1\right)y'(t_n)$$

$$+ h^2\left(\frac{1}{2} - \frac{3}{2} + 1\right)y''(t_n) + h^3\left(\frac{1}{6} + \frac{1}{2} - \frac{2}{3}\right)y'''(t_n) + O(h^4)$$

$$- 3hf(t_n, y_n)$$

$$\Rightarrow \cancel{3hy'(t_n)} + h^3\underbrace{\left(\frac{1}{6} + \frac{3}{6} - \frac{4}{6}\right)}_{=0}y'''(t_n) + O(h^4) - \cancel{3hf(t_n, y_n)} =$$

$\Rightarrow \quad \mathcal{L}\{y(t)\} =$

Thus this method is at least $O(h^4)$ checking [illegible] a typo in the text book.

$O(h^4)$ term is

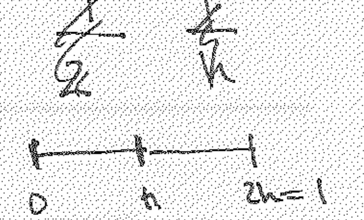
$$h^4\left[\frac{1}{4!}+\frac{3}{4!}+\frac{1}{2}\frac{2^4}{4!}\right]y^{IV}(t_n) = \frac{h^4}{4!}\underbrace{(4+2^3)}\,y^{IV}(t_n)$$

eq 2.29 $\quad y_{n+1}+\frac{3}{2}y_n-3y_{n-1}+\frac{1}{2}y_{n-2}-3hf(t_n,y_n)=0$

eq 2.30 $\quad y_{n+1}-y_n-h\left[\quad \cdots \quad\right]=0$

w/ $y'=-y \qquad f(t,y)=-y \qquad$ w/ $h_i=2^{-i} \qquad nstep=\frac{1}{h}=\frac{1}{2^{-i}}=2^i$

$y(0)=1 \qquad t\in[0,1]$



w/ $y(t)=e^{-t}$

eq 2.29 is

$$y_{n+1} = -\frac{3}{2}y_n+3y_{n-1}-\frac{1}{2}y_{n-2}+3hf(t_n,y_n)$$

$$= \quad '' \qquad '' \qquad '' \quad -3hy_n$$

$$= -\left(\frac{3}{2}+3h\right)y_n+3y_{n-1}-\frac{1}{2}y_{n-2}$$

$h=1/2$

$y_0=1$

$y_{-1}=e^{-h}$

$y_{-2}=e^{-2h}$

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(29) Backwards Euler:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$ considering absolute ~~stab~~ stability

we consider stability w.r.t. the scalar test equation $w' = \lambda w$.

Then the backwards Euler method becomes

$$y_{n+1} = y_n + h\lambda y_{n+1}$$

$$\Rightarrow (1-h\lambda)\, y_{n+1} = y_n \quad \Rightarrow \quad y_{n+1} = \left(\frac{1}{1-h\lambda}\right) y_n$$

For absolute stability, we require that $\left|\frac{1}{1-h\lambda}\right| < 1$

$\iff 1 < |1-h\lambda|$ Assuming $\lambda < 0$ so the test eq

$\iff$ $w' = \lambda w$ is itself bounded + the equation is stable.

Then the set $\cancel{S = \{\, | 1+z | > 1, \, Re(z) \}}$

$\cancel{S = \{ \frac{1}{|1-h\lambda|}}$ $\quad S = \left\{ \frac{1}{|1-\cancel{h\lambda} z|} < 1, \; Re(z) < 0 \right\}$

= ~~right~~ left half of the complex plane as stated

(b) trapezoidal rule

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

Again under the guise of absolute stability ~~let~~ ~~if~~ consider $w' = \lambda w$.

then the trapezoidal rule becomes

$$y_{n+1} = y_n + \frac{h}{2}(\lambda y_n + \lambda y_{n+1}) \iff \left(1 - \frac{h\lambda}{2}\right) y_{n+1} = \left(1 + \frac{h\lambda}{2}\right) y_n$$

$$\iff y_{n+1} = \frac{\left(1 + \frac{h\lambda}{2}\right)}{\left(1 - \frac{h\lambda}{2}\right)} y_n$$

Stability of this difference eq requires $\dfrac{\left|1 + \frac{h\lambda}{2}\right|}{\left|1 - \frac{h\lambda}{2}\right|} < 1$

let $z \equiv h\lambda$ then $\forall\ \lambda < 0$

$$S = \left\{ \frac{|1 + z/2|}{|1 - z/2|} < 1,\ \mathrm{Re}(z) < 0 \right\}$$ corresponds to the left

half of the complex plane

(c) Heun's method

$$y_{n,1} = y_n + h f(t_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n,1})\right)$$

Then again considering absolute stability, we will investigate the stability of this method on the eq $w' = \lambda w \qquad \lambda < 0$

$$y_{n,1} = y_n + h\lambda y_n = (1+h\lambda) y_n$$

$$y_{n+1} = y_n + \frac{h}{2}(\lambda y_n + \lambda y_{n,1}) = y_n + \frac{h}{2}(\lambda y_n + (1+h\lambda)\lambda y_n)$$

$$y_{n+1} = y_n\left[1 + \frac{h}{2}\lambda + \lambda\frac{h}{2} + \lambda^2\frac{h^2}{2}\right] = \frac{y_n}{2}\left[2 + 2\lambda h + \lambda^2 h^2\right]$$

Then stability of this difference eq requires

$$\left|\frac{2 + 2\lambda h + \lambda^2 h^2}{2}\right| < 1$$

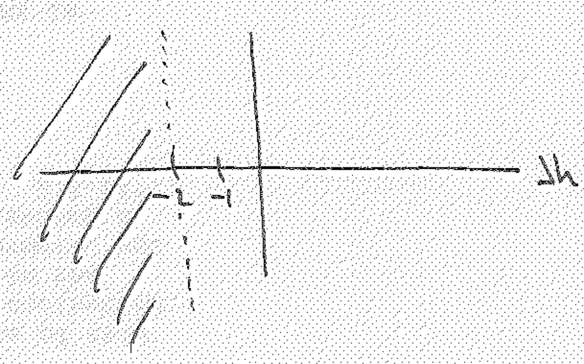
$\iff \quad |2 + 2\lambda h + \lambda^2 h^2| < 2 \qquad$ that this stability region for fixed $\lambda < 0$ & $h > 0$ is finite can be seen by observing that

$$\iff \qquad -2 < 2 + 2\lambda h + \lambda^2 h^2 < 2$$

$\iff$

$0 < 4 + 2\lambda h + \lambda^2 h^2 \quad \& \quad 2\lambda h + \lambda^2 h^2 < 0$

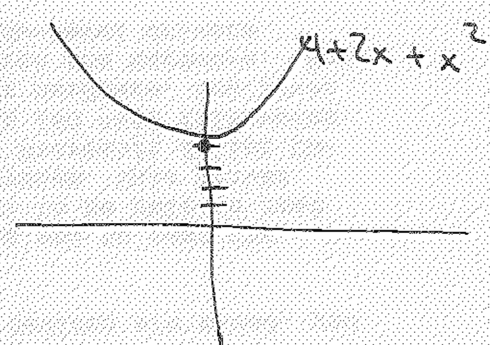
$\lambda h = \dfrac{-2 \pm \sqrt{4 - 4(1)(4)}}{2(1)}$ $\Rightarrow$ $\lambda h = 0$ or $\lambda h < -2$



$= \dfrac{-2 \pm 2\sqrt{1-4}}{2}$

$= -1 \pm 3i$

if $\lambda h > -1 \pm 3i$



seems that if ~~[illegible]~~

$\lambda h < -2$ Heun's method will be stable... in contradiction

to that discussed in the text...?

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2.10

pt (c) $\quad y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y \qquad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$f(y) = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}$$

$$\frac{\partial f}{\partial y} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \checkmark$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right]$$

$$= \underbrace{\tfrac{h}{2}}_{h\gamma} f(t_{n+1}, y_{n+1}) + \underbrace{y_n + \tfrac{h}{2} f(t_n, y_n)}_{\equiv\, \Upsilon}$$

$\equiv \Upsilon$ on pg 69 Shampine.

$h\gamma$

w/ $\gamma = \frac{1}{2}$

Then Newton iterations ~~are~~ are:

$$y_{n+1}^{[0]} = y_n$$

$$\cancel{\left(I - \tfrac{h}{2} J\right)\Delta_m = y_n + \tfrac{h}{2} f(t_{n+1}, y_{n+1}) - y_{n+1}^{[m]}}$$

$$\cancel{y_{n+1}^{[m]} = y_{n+1}^{[m]} + \Delta_m}$$

$$(I - h\gamma J)\Delta_m = \Upsilon + \cancel{y_n + f(t_{n+1}, y_{n+1}^{[m]}) - y_{n+1}^{[m]}}$$

$$= \cancel{\tfrac{h}{2} f(t_{n+1}, y_{n+1}^{[m]}) - y_{n+1}^{[m]}}$$

$$y_{n+1}^{[m+1]} = y_{n+1}^{[m]} + \Delta_m$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_{n+1},y_{n+1})\right)$$

$$y_{n+1} \equiv y_{n+1}^{[m]} + \Delta_m.$$

$$y_{n+1}^{[m]} + \Delta_m = y_n + \frac{h}{2} f(t_n,y_n) + \underbrace{\frac{h}{2} f(t_{n+1}, y_{n+1}^{[m]} + \Delta_m)}_{\frac{h}{2}\left[f(t_{n+1}, y_{n+1}^{[m]}) + \left(\frac{\partial f}{\partial y_{n+1}}\right)\Big|_{y_{n+1}^{[m]}} \Delta_m \right]}$$

$$\left(\frac{\partial f}{\partial y_{n+1}}\right)\Big|_{y_{n+1}^{[m]}} \equiv J$$

$$\Rightarrow \boxed{\left[I - J\frac{h}{2} \right] \Delta_m = y_n + \frac{h}{2} f(t_n, y_n) + \frac{h}{2} f(t_{n+1}, y_{n+1}^{[m]}) - y_{n+1}^{[m]}}$$

From ~~befor~~ general method on pg 69

$$y_{n+1}^{[0]} = y_n$$

$$\left(I - \frac{h}{2}J\right)\Delta_m = y_n + \frac{h}{2} f(t_n,y_n) + \frac{h}{2} f(t_{n+1}, y_{n+1}^{[m]}) - y_{n+1}^{[m]}$$

$$y_{n+1}^{[m+1]} = y_{n+1}^{[m]} + \Delta_m$$

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(211) Using AB$\rlap{1}$, we ~~xxx~~ evaluate the local solution

$$u' = f(t,u) \qquad u(t_n) = y_n$$

on $$u(t_n+h) = y_n + \int_{t_n}^{t_n+h} f(x,u(x))\,dx$$ by ~~xxx~~ interpolating $f(t,u(t))$ ~~xx~~ ~~xx~~ ~~xxx~~ at

s pts w/ $f_{n,j} = f(t_{n,j}, y_{n,j})$ & integrating ~~x~~ this interpolating polynomial..

$AB_n \Rightarrow$ let $t_{n,j} = t_{n-j}$ & $y_{n,j} = y_{n-j}$

So AB1 is $f(x,u(x)) \cong f(t_{n-0}, y_{n-0})$

So

with $y_{n+1} = y_n + h f(t_{n}, y_{n})$ in the problem given $f(t,y) = -100y + 10$

~~xxx~~ $\mathcal{L}\{y\} \equiv y_{n+1} - y_n - h(-100\,y_n + 10)$

$$= y_{n+1} + \cancel{xxx}\,(-1 + 100h)\,y_n - 10h$$

$$= \cancel{y_n} + h y_n' + \frac{h^2}{2} y_n'' + O(h^3) + (\cancel{-1} + 100h)\,y_n - 10h \qquad \text{w/ } y_n' = -100 y_n + 10$$

$$= h(-\cancel{100}y_n \cancel{+10}) + \frac{h^2}{2} y_n'' + O(h^3) + \cancel{100h}y_n - \cancel{10h}$$

$$= \frac{h^2}{2} y_n'' + O(h^3)$$

since $y' = -100y + 10$

$y'' = -100y' = \cancel{-10^4 y + 10^3}$

$= -100(-100y + 10) = 10^4 y - 10^3$

$\mathcal{L}\{\cancel{y(t)}\} =$

$$le_n = \frac{h^2}{2}(10^4 y - 10^3)$$

if we require $|le_n| \leq \tau$

$$\Leftrightarrow \quad 10^3 h^2 |10y - 1| \leq 2\tau.$$

$$\Leftrightarrow \quad h \leq \left(\frac{2\tau}{10^3 |10y-1|}\right)^{1/2} = \frac{\sqrt{2}\,\tau^{1/2}}{10^{3/2}\sqrt{|10y-1|}}$$

~~small~~ The true solution to this problem is

$$\left\{ y' = -100y + 10 \qquad y(0) = 1 \right\} \qquad y(t) = \frac{1}{10} + \frac{9}{10}e^{-100t} > \frac{1}{10}$$

Then from the above inequality $10y - 1 \cong 9e^{-100t}$

$$h \leq \frac{\sqrt{2}\,\tau^{1/2}}{10^{3/2}\, 3 e^{-50t}} = C_1 \cdot \tau^{1/2} e^{50t}$$

So for $t \cong 0$ $\quad h_0 \cong C_1 \tau^{1/2}$

with $t \cong 1$ $\quad h_1 \cong C_1 \tau^{1/2} e^{50}$ $\quad$ so $h_1 \gg h_0$ for

errors under threshold τ.

(2.12) $y' = 10y \qquad y(0) = 1$

has true solution $\quad y(t) = e^{10t}$

with the backwards Euler method:

$$y_{n+1} = y_n + hf(y_{n+1}) = y_n + h(10)y_{n+1} \quad \checkmark$$

$$\Rightarrow \quad (1 - 10h)\, y_{n+1} = y_n$$

$$y_{n+1} = \left(\frac{1}{1-10h}\right) y_n \quad \checkmark$$

Then as long as $|1-10h| > 1$ this method is stable, but for large h's may not compute an accurate solution. With $h=1$ we have

$$y_{n+1} = \frac{-1}{9} y_n \qquad y_0 = 1$$

$\therefore \; y_n = \left(\frac{-1}{9}\right)^n$ An oscillatory solution decaying to zero when the true solution blows up. $y(t \to \infty) = +\infty$, ~~This method is ...~~

2.13

$$y' = -\frac{1}{t^2} + 10\left(y - \frac{1}{t}\right) \qquad y(1) = 1$$

If we expect this will decay to zero, might use backwards Euler

Note that on equation of this type can be constructed by analytically solving $y'(t) = f$ for say Y

& then solving for the solution to $y' = f + G(y - Y)$.

using ode45 & ode15s

pg 78-79 shampine

(2.15)

$$y' = y^2(1-y)$$

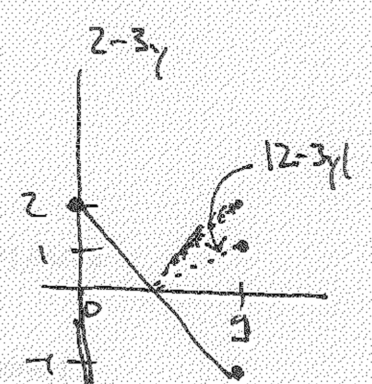
$$\frac{\partial f}{\partial y} = 2y(1-y) + y^2(-1) = 2y - 2y^2 - y^2 = 2y - 3y^2 = y(2-3y)$$

A Lipschitz constant L is one such that

$$\|f(t,u) - f(t,v)\| \le L\|u - v\|$$

if $\left|\frac{\partial f}{\partial y}\right| < L$ in a region R f will satisfy a Lipschitz's condition there.

$$\left|\frac{\partial f}{\partial y}\right| = |y(2-3y)| \le |2-3y| \le 2$$



For the 1st part of the integration y will around

$t \sim O(1/\epsilon)$ $\quad y \approx 0 \quad \therefore \quad \frac{\partial f}{\partial y} \cong 2$ by linear stability analysis

we expect this IVP to be linearly unstable (since $2 > 0$) & $\therefore$ not stiff?

over the small region of rapid change (9600 – 9700) $y \in [0,1]$

& $\frac{\partial f}{\partial y}$ changes from 2 to -1

Don't see how integration interval influences the stiffness of the IVP??

2.16 $y' = \lambda y$ by BO Shampine

$$\sum_{i=0}^{k} \alpha_i y_{n+1-i} - h \sum_{i=0}^{k} \beta_i (\lambda y_{n+1-i}) = \sum_{i=0}^{k} (\alpha_i - h\lambda\beta_i) y_{n+1-i} = 0$$

~~$\alpha_0 y_{n+1} + \alpha_1 y_n + \alpha_2 y_{n-1} + \cdots + \alpha_k y_{n+1-k} - h$~~

$$\alpha_0 y_{n+1} + \alpha_1 y_n + \alpha_2 y_{n-1} + \cdots + \alpha_k y_{n+1-k} - \lambda h[\beta_0 y_{n+1} + \beta_1 y_n + \beta_2 y_{n-1} + \cdots + \beta_k y_{n+1-k}] = 0$$

let $y_n \propto r^n$ then LMM becomes

$$\sum_{i=0}^{k} (\alpha_i - h\lambda\beta_i) r^{n+1-i} = 0 \qquad \text{Mult by } r^{-n-1+k}$$

$$\Leftrightarrow \quad \sum_{i=0}^{k} (\alpha_i - \lambda h\beta_i) r^{k-i} = 0$$

let $h\lambda = z$

$$\underbrace{\sum_{i=0}^{k} \alpha_i r^{k-i}}_{\equiv\ \rho(r)} - z \underbrace{\sum_{i=0}^{k} \beta_i r^{k-i}}_{\equiv\ \sigma(r)} = 0$$

$$\Rightarrow \quad z = \frac{\rho(r)}{\sigma(r)} \quad \text{for} \quad r = e^{i\theta} \quad \theta \in [0, 2\pi]$$

For Forward Euler LMM

$$y_{n+1} = y_n + hf(y_n)$$

$$\Rightarrow \quad y_{n+1} = y_n + h\lambda y_n$$

$$\Rightarrow \quad y_{n+1} - y_n - h\lambda y_n = 0$$

$$\Rightarrow \quad \alpha_0 = 1 \; ; \; \alpha_1 = -1 \; ; \; \beta_0 = 0, \; \beta_1 = -1$$

So

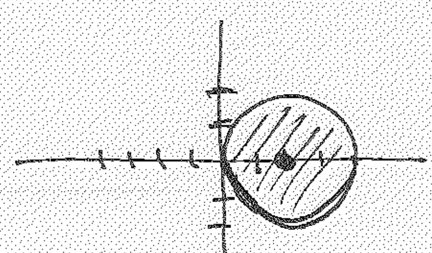
$$\rho(r) = \sum_{i=0}^{1} \alpha_i r^{1-i} = \alpha_0 r + \alpha_1 = r - 1$$

&

$$\sigma(r) = \sum_{i=0}^{1} \beta_i r^{1-i} = \beta_0 r + \beta_1 = -1$$

$$\therefore \quad z = \frac{r-1}{-1} = 1 - r$$

For $r = e^{i\theta}$ gives $z(\theta) = 1 - e^{i\theta}$



let $\theta = \text{linspace}(0, 2\pi, 100)$

$z = 1 - e^{i\theta}$ $\qquad$ $x = \text{real}(z)$

$\qquad\qquad\qquad\qquad y = \text{imag}(z)$

plot (x, y)

$$z = 1 - (\cos\theta + i\sin\theta) = 1 - \cos\theta - i\sin\theta$$

$x = 1 - \cos\theta$ $\Rightarrow$ eq for a circle

$y = \cancel{\text{tsb}}\ -\sin\theta$

* <u>Backward Euler method:</u>

$y_{n+1} = y_n + h f(y_{n+1}) \Rightarrow y_{n+1} - y_n = h\lambda y_{n+1}$ w/ $f(y) = \lambda y$

let $y_n = r^n$ + $z = h\lambda$

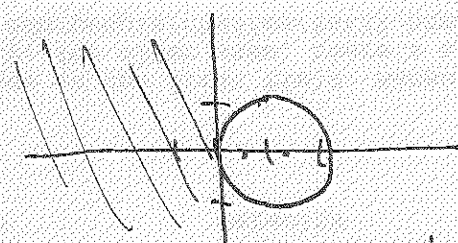
$$r^{n+1} - r^n = z r^{n+1} \Rightarrow \frac{r-1}{r} = z$$

let $r = e^{i\theta}$ then $z = 1 - 1/r = 1 - e^{-i\theta}$ or $\frac{1}{r} = 1 - z$

$r = \frac{1}{1-z}$

$x = 1 - \cos\theta$ + $y = \sin\theta$

$(1-x)^2 + y^2 = 1$



~~cases~~ to determine if the stable region is inside of or outside of this circle consider a point inside the circle, say $z = +1$

~~then~~ + determine if $|r| \overset{?}{\gtrless} 1$ for $z = 1+\epsilon$ the eq for r is

$r - 1 = r$

— The trapezoid rule is

$$y_{n+1} = y_n + \frac{h}{2}\left(f(y_n) + f(y_{n+1})\right)$$

to consider ~~[illegible]~~ the root locus method let $f(y) = \lambda y$ to get

$$y_{n+1} = y_n + \frac{h}{2}\lambda y_n + \frac{h}{2}\lambda y_{n+1}$$

This is a constant coeff difference eq & is solved w/ $y_n = r^n$

$\div$ by r^n

$$r = 1 + \frac{h\lambda}{2} + \frac{h\lambda}{2} r$$

$$r - 1 = \frac{1}{2}(1+r)z \Rightarrow z = \frac{2(r-1)}{1+r}$$

let $r = e^{i\theta}$ then $z = \dfrac{2(\cos\theta - 1 + i\sin\theta)}{1+\cos\theta + i\sin\theta}$

$$\Rightarrow z = \frac{2(\cos\theta - 1 + i\sin\theta)(1+\cos\theta - i\sin\theta)}{(1+\cos\theta + i\sin\theta)(1+\cos\theta - i\sin\theta)}$$

$$=$$

equate ~~[illegible]~~ real & imag part to x & y & plot v.s. θ.

or eliminate θ to get cartesian representation.

$$z = \frac{2(e^{i\theta} - 1)}{1 + e^{i\theta}} \cdot \frac{e^{-i\theta/2}}{e^{-i\theta/2}} = \frac{2(e^{i\theta/2} - e^{-i\theta/2})}{e^{i\theta/2} - e^{-i\theta/2}} = 2\left(\frac{\cosh(\theta/2)}{\sinh(\theta/2)}\right)$$

The plots of this look funny?...

BDF2: with constant step size we get

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = hf(t_{n+1}, y_{n+1})$$

For absolute stability analysis consider $f(t,y) = \lambda y$ we get

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = h\lambda y_{n+1}$$

to solve this difference equation $y_n \propto r^n$ w/ r satisfying a characteristic equation

$$\frac{3}{2}r - 2 + \frac{1}{2}r^{-1} = h\lambda r$$

$\Leftrightarrow$ $\frac{3}{2}r^2 - 2r + \frac{1}{2} = h\lambda r^2$ The stability region for this problem is

$\Leftrightarrow$ determined when $|r| = 1$

$\Leftrightarrow$ $h\lambda = \dfrac{\frac{3}{2}r^2 - 2r + \frac{1}{2}}{r^2}$ let $z = h\lambda$

Then $z = \left.\dfrac{\frac{3}{2}r^2 - 2r + \frac{1}{2}}{r^2}\right|_{r=e^{i\theta}}$

$$z = \frac{\frac{3}{2}e^{2i\theta} - 2e^{i\theta} + \frac{1}{2}}{e^{2i\theta}}$$

(1) Expand $e^{i\theta}$ in terms of $\cos\theta + i\sin\theta$

(2) Expand $z = x + iy$

(3) Equate real + imaginary parts

(4) Eliminate θ to get an expression for x in terms of y

AM3: Determine the stability of AM3 using the root locus method

$$y_{n+1} = y_n + h\left[\frac{5}{12}y'_{n+1} + \frac{8}{12}y'_n - \frac{1}{12}y'_{n-1}\right]$$

$$\Rightarrow \quad y_{n+1} = y_n + h\left[\frac{5}{12}f(y_{n+1}) + \cdots \right]$$ y's in terms of f.

For absolute stability let $f(t,y) = \lambda y$ then AM3 becomes

$$y_{n+1} = y_n + h\lambda\left[\frac{5}{12}y_{n+1} + \frac{8}{12}y_n - \frac{1}{12}y_{n-1}\right]$$

Then let $y_n = r^n$ to obtain

$$z = h\lambda = \frac{y_{n+1} - y_n}{\left[\frac{5}{12}y_{n+1} + \cdots\right]} = \frac{r^{n+1} - r^n}{\left[\frac{5}{12}r^{n+1} + \frac{8}{12}r^n - \frac{1}{12}r^{n-1}\right]}$$

$$h\lambda = \frac{r^2 - 1}{\left(\frac{5}{12}r^2 + \frac{8}{12}r - \frac{1}{12}\right)}$$ where $|r| = 1$ $r = e^{i\theta}$

So $$z = \frac{e^{2i\theta} - 1}{\left(\frac{5}{12}e^{2i\theta} + \frac{8}{12}e^{i\theta} - \frac{1}{12}\right)}$$

BDF3:

$$y_{n+1} = \frac{18}{11} y_n - \frac{9}{11} y_{n-1} + \frac{2}{11} y_{n-2} + \frac{6}{11} h\, y'_{n+1}$$

w/ Absolute stability, $y' = f(y) = \lambda y$ so BDF3 becomes,

$$y_{n+1} = \frac{18}{11} y_n - \frac{9}{11} y_{n-1} + \frac{2}{11} y_{n-2} + \frac{6}{11} (h\lambda)\, y_{n+1}$$

let $y_n = r^n$

$$\Rightarrow \; z = \lambda h = \frac{r - \frac{18}{11} - \frac{9}{11} r^{-1} - \frac{2}{11} r^{-2}}{\frac{6}{11} r}$$

w/ $r = e^{i\theta}$

$$z = \frac{e^{i\theta} - \frac{18}{11} - \frac{9}{11} e^{-i\theta} - \frac{2}{11} e^{-2i\theta}}{\frac{6}{11} e^{i\theta}} \quad \ldots$$

(2.22)

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Let $y_1(t) = x(t)$

$y_2(t) = x'(t)$

Then $y_1'(t) = x'(t) = y_2(t)$

$$y_2'(t) = x''(t) = -2\zeta x'(t) - x - 1.5x^2(t) - .5x^3(t) + F_2\cos(\Omega_2 t) + F_3\cos(\Omega_3 t)$$

$$= -2\zeta y_2 - y_1 - 1.5y_1^2 - .5y_1^3 + F_2\cos(\Omega_2 t) + F_3\cos(\Omega_3 t)$$

$y_1' = y_2$

$y_2' = -2\zeta y_2 - y_1 - 1.5y_1^2 - .5y_1^3 + F_2\cos(\Omega_2 t) + F_3\cos(\Omega_3 t)$

1

pg ~~[illegible]~~ 102 Shampine

2.23

let ~~$y_1 = x(t)$~~ $y_1(t) = x(t)$

~~$y_2(t) = x'(t)$~~ $y_2(t) = x'(t)$

$$y_1'(t) = x'(t) = y_2(t)$$

$$y_2'(t) = x''(t) = g(1+q)\left[\left(1+\frac{x}{d}\right)^{-r} + \frac{Rt}{100} - 1 + \frac{qx}{L(1+q)}\right]$$

$$= g(1+q)\left[\left(1+\frac{y_1}{d}\right)^{-r} + \frac{Rt}{100} - 1 + \frac{qy_1}{L(1+q)}\right]$$

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} y_2(t) \\ g(1+q)\left[\left(1+\frac{y_1(t)}{d}\right)^{-r} + \frac{Rt}{100} - 1 + \frac{q\,y_1(t)}{L(1+q)}\right] \end{pmatrix}$$

(2.24) $\begin{pmatrix} y' \\ v' \\ \phi' \end{pmatrix} = \begin{pmatrix} \tan(\phi) \\ \dfrac{-g\sin(\phi) + bv^2}{v\cos(\phi)} \\ -\dfrac{g}{v^2} \end{pmatrix}$ pg 103 Schenzel

(2.25) $\theta'' + \sin(\theta) = 0$

let $y_1 = \theta$
$y_2 = \theta'$

$y_1' = y_2$

$y_2' = -\sin(y_1)$

(2.26) $t(x) = \displaystyle\int_{x_0}^{x} \frac{1}{\sqrt{f(s)}}\, ds$

$\dfrac{dt}{dx} = \dfrac{1}{\sqrt{f(x)}}$ $\quad t(x_0) = 0$

$\Longleftrightarrow \quad \dfrac{dx}{dt} = \sqrt{f(x)} \quad\quad x(0) = x_0$

$\dfrac{d^2x}{dt^2} = \dfrac{1}{2} f(x)^{-1/2} f'(x)\, x'(t)$

$= \dfrac{1}{2} \dfrac{1}{\cancel{x'(t)}} f'(x)\, \cancel{x'(t)} \quad\quad x(0) = x_0\ ; \ x'(0) = \sqrt{f(x_0)}$

$= .5\, f'(x)$

2.26

- $f = \cancel{}\ x$

$$t(x) = \int_0^x \frac{1}{\sqrt{xs}}\,ds$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{f(x)}} = \frac{1}{\sqrt{x}}$$

$$dt = \frac{1}{\sqrt{x}}dx = x^{-1/2}dx$$

$$t - t_0 = \frac{x^{1/2}}{(1/2)}\Bigg|_{x_0}^{x} = 2(x^{1/2} - x_0^{1/2})$$

$$x^{1/2} = x_0^{1/2} + \frac{(t-t_0)}{2}$$

if ~~xxxxxxx~~ $t_0 = 0 \quad x(0)^{1/2} = x_0^{1/2} = 0 \Rightarrow x_0 = 0$

$$x^{1/2} = \frac{t}{2} \Rightarrow t = 2\sqrt{x}.$$

- $f = 1 - x$

$$dt = \frac{1}{\sqrt{1-x}}dx$$

$$t - t_0 = 2(-1)(1-x)^{1/2}\Bigg|_{x_0}^{x} = 2(-1)(1-x)^{1/2} + 2(1-x_0)^{1/2}$$

$t_0 = 0 \quad x(t_0) = 0$

$\Rightarrow \quad t = (-2)(1-x)^{1/2} + 2$

- $f = 1 - x^2$

$dt = \dfrac{1}{\sqrt{1-x^2}}\, dx$

$t - t_0 = \arcsin(x) - \arcsin(x_0)$

$\Rightarrow t = \arcsin(x)$

For $f(x) = x$ the ODE to solve is:

if $f(x) = x \quad f'(x) = 1$ & DE is

$x'' = .5 \qquad x(0) = 0;\ x'(0) = 0 \quad \Rightarrow \quad x(t) = \dfrac{.5t^2}{2} = .25t^2$

$\Rightarrow \dfrac{d}{dt}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x' \\ .5 \end{pmatrix}$ $\qquad$ or $t = 2\sqrt{x}$

For $f(x) = 1 - x$ the ODE to solve is:

$f'(x) = -1$

$\therefore \ x'' = -.5$ w/ I.C. $x(0) = 0 \quad x'(0) = \sqrt{1} = 1$

$\Rightarrow \dfrac{d}{dt}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x' \\ -.5 \end{pmatrix}$ w/ I.C. $\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $f(x) = 1 - x^2$ the ODE to solve is:

$$f'(x) = -2x$$

$$\therefore \quad x'' = .5(-2x) = -x \qquad \text{w/ I.C.} \quad x(0) = 0 \,; \; x'(0) = \sqrt{f(x(0))} = 1$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}$$

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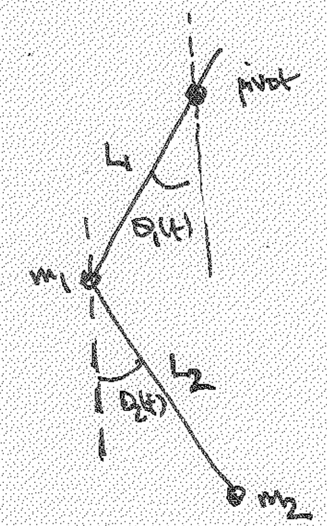
(2.33)



$m_1 = 937.5$ slugs

$m_2 = 312.5$ slugs

$L_1 = L_2 = 16$ ft

$g = 32$ ft/s

if $|\theta_2(t)| > \pi/3$

let $$Y = \begin{pmatrix} \theta_1(t) \\ \theta_1'(t) \\ \theta_2(t) \\ \theta_2'(t) \end{pmatrix}$$ then $$Y' = \begin{pmatrix} \theta_1'(t) \\ \theta_1''(t) \\ \theta_2'(t) \\ \theta_2''(t) \end{pmatrix}$$

is such that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (m_1+m_2)L_1 & 0 & m_2 L_2 \cos(\theta_2-\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & m_2 L_1 \cos(\theta_2-\theta_1) & 0 & m_2 L_2 \end{bmatrix} \begin{bmatrix} \theta_1' \\ \theta_1'' \\ \theta_2' \\ \theta_2'' \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (m_1+m_2)L_1 & 0 & m_2 L_2 \cos(\theta_2-\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & m_2 L_1 \cos(\theta_2-\theta_1) & 0 & m_2 L_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \theta_1 \\ \theta_1' \\ \theta_2 \\ \theta_2' \end{bmatrix}$$

$$= \begin{bmatrix} \theta_1' \\ m_2 L_2 (\theta_2')^2 \sin(\theta_2-\theta_1) - (m_1+m_2) g \sin(\theta_1) \\ \theta_2' \\ -m_2 L_1 (\theta_1')^2 \sin(\theta_2-\theta_1) - m_2 g \sin\theta_2 \end{bmatrix}$$

in terms of $y_1(t), \ldots, y_4(t)$ we get

$$\cancel{\begin{bmatrix} 1 & & 0 \\ 0 & (m_1+m_2)L_1 & \\ 0 & & 0 \\ 0 & m_2 L_1 \cos(\theta_2-\theta_1) & \end{bmatrix}}$$

$y_1 = \theta_1$
$y_2 = \theta_1'$
$y_3 = \theta_2$
$y_4 = \theta_2'$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (m_1+m_2)L_1 & 0 & m_2 L_2 \cos(y_3 - y_1) \\ 0 & 0 & 1 & 0 \\ 0 & m_2 L_1 \cos(y_3 - y_1) & 0 & m_2 L_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= \begin{bmatrix} y_2 \\ m_2 L_2 y_4^2 \sin(y_3 - y_1) - (m_1+m_2) g \sin(y_1) \\ y_4 \\ -m_2 L_1 y_2^2 \sin(y_3 - y_1) - m_2 g \sin(y_3) \end{bmatrix}$$

w/ ic of $\quad \underline{y}(0) = \begin{bmatrix} -.5 \\ -1 \\ 1 \\ 2 \end{bmatrix}$

We want to see if $|\theta_2(t)| > \frac{\pi}{3}$ at any point. If so the 2nd pellet will disconnect, thus we need an event function to look for this event.

for the linear model let $y = \begin{pmatrix} \theta_1 \\ \theta_1' \\ \theta_2 \\ \theta_2' \end{pmatrix}$

then $\dfrac{dy}{dt} = \begin{pmatrix} \theta_1' \\ \theta_1'' \\ \theta_2' \\ \theta_2'' \end{pmatrix}$

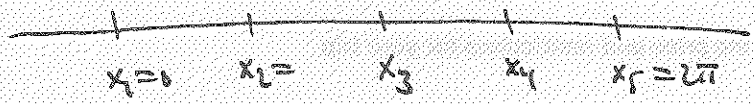
So the Diff eq (in mass matrix form) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \cancel{(m_1+m_2)L_1}\ 0 & (m_1+m_2)L_1 & 0 & m_2L_2 \\ 0 & 0 & 1 & 0 \\ 0 & m_2L_1 & 0 & m_2L_2 \end{bmatrix} \frac{d}{dt}\begin{pmatrix} \theta_1 \\ \theta_1' \\ \theta_2 \\ \theta_2' \end{pmatrix} = \begin{bmatrix} \theta_1' \\ -(m_1+m_2)g\,\theta_1 \\ \theta_2' \\ -m_2 g\,\theta_2 \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (m_1+m_2)L_1 & 0 & m_2L_2 \\ 0 & 0 & 1 & 0 \\ 0 & m_2L_1 & 0 & m_2L_2 \end{bmatrix} \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_2 \\ -(m_1+m_2)g\,y_1 \\ y_4 \\ -m_2 g\,y_3 \end{bmatrix}$$

(2.38) Let

$N = 5$



Then the system of ODE's is :

$$\frac{dv_1}{dt} = -c(x_1)\left(\frac{v_1 - v_0}{h}\right) \cong 0$$ (case considered in the text or $-c(x_1)\left(\frac{v_1 - v_5}{h}\right)$ for periodic B.C.s.

$$\frac{dv_2}{dt} = -c(x_2)\left(\frac{v_2 - v_1}{h}\right)$$

$$\frac{dv_3}{dt} = -c(x_3)\left(\frac{v_3 - v_2}{h}\right)$$

$$\frac{dv_4}{dt} = -c(x_4)\left(\frac{v_4 - v_3}{h}\right)$$

$$\frac{dv_5}{dt} = -c(x_5)\left(\frac{v_5 - v_4}{h}\right)$$

Then $\cancel{J =}$

$$M = \begin{bmatrix} -\frac{c_1}{h} & 0 & 0 & 0 & \frac{c_1}{h} \\ +\frac{c_2}{h} & -\frac{c_2}{h} & 0 & 0 & 0 \\ 0 & +\frac{c_3}{h} & -\frac{c_3}{h} & 0 & 0 \\ 0 & 0 & \frac{c_4}{h} & -\frac{c_4}{h} & 0 \\ 0 & 0 & 0 & \frac{c_5}{h} & -\frac{c_5}{h} \end{bmatrix}$$

is the mass coefficient, but since we are asked about computing the jacobian we see that

(2.3b) $u_t = u_{xx} + g(u)$ $\qquad u(0,t) = 0$

$u_x(1,t) = 0$

$$g(u) = \begin{cases} -Au & u \le u_c \\ 0 & u_c < u \end{cases}$$

Let the grid be $x_m = mh$ $\quad m = 0, 1, 2, \ldots, N+1$ w/ $h \equiv \frac{1}{N}$

then $v_m(t) \triangleq u(x_m, t)$

$$\frac{dv_m}{dt} = \frac{v_{m+1} - 2v_m + v_{m-1}}{h^2} + g(v_m)$$

when ~~x~~ $m = 1$

$$\frac{dv_1}{dt} = \frac{v_2 - 2v_1 + v_0}{h^2} + g(v_1)$$ + thus requires $v_0(t)$ to satisfy

$u(0,t) = 0 \quad \forall t$ define $v_0(t) = 0$

Then $$\frac{dv_1}{dt} = \frac{v_2 - 2v_1}{h^2} + g(v_1)$$

when $m = N$ we get

$$\frac{dv_N}{dt} = \frac{v_{N+1} - 2v_N + v_{N-1}}{h^2} + g(v_N)$$ requiring $v_{N+1}(t)$

to satisfy $v_x(1,t) = 0 \quad \forall t$ we will ~~enforce~~ $\cancel{v_{N+1} = v_N \ \forall t}$

approximated v_x w/ central differences $\frac{v_{N+1} - v_{N-1}}{2h} \doteq 0$

Then $v_{N+1} = v_{N-1}$ thus the eq for $v_N(t)$ becomes

$$\frac{dv_N}{dt} = \frac{-2v_N + 2v_{N-1}}{h^2} + g(v_N)$$

Now I.C. for all of $v_m(t)$ are given by $v_m(0) = u(x_m, 0)$

Thus the system of ODE's to solve is

$$\frac{dv_1}{dt} = \frac{v_2 - 2v_1}{h^2} + g(v_1)$$

$$\frac{dv_m}{dt} = \frac{v_{m+1} - 2v_m + v_{m-1}}{h^2} + g(v_m) \qquad m = 2, 3, \cdots, N-1$$

$$+ \quad \frac{dv_N}{dt} = \frac{-2v_N + 2v_{N-1}}{h^2} + g(v_N)$$

~~I don't think that~~ the Jacobian cannot be evaluated explicitly since the ~~R~~ functional form of $g(\cdot)$ depends on the magnitude of u, but one can certainly ~~supply~~ supply the sparsity pattern of the Jacobian... I'll assume that there is a typo in the definition of $g(\cdot)$. I'll assume that the correct definition is

$$g(u) = \begin{cases} -A & u \le u_c \\ 0 & u > u_c \end{cases} \qquad /* \text{ No } u */$$

Then the jacobian is constant as indicated in the text e = ones(N,1)

Then ~~J = spdiags([e~~

```
J = spdiags( [e, -2e, e], -1:+1, N,N)/h^2;
J(1,3) = 0 ;  ✓
J(N,N-1) = 2/h^2 ;

options = odeset('Jacobian', J);
[t, v] = ode15s( @f, tspan, v0, options );
% Add the BV's:
x = [0 x 1];

x = h*(1:N)
v0 =

dvdt = f(t,v)
global J h;
dvdt = J * v + g(v)     ...   see matlab file for complete
```

Solution.

(2.37) Pg 130 Shampini

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^n = 0 \quad *$$

$y'(0) = 0$ & $y(0) = 1$; ~~xxxxxx~~ Physically we expect a smooth solution at $x=0$.

Thus looking for a taylor series for $y(x)$ at $x \approx 0$ gives

$$y(x) = \sum_{n \geq 0} y_n x^n \qquad y(0) = 1 \Rightarrow y_0 = 1$$

$$y'(0) = 0 \Rightarrow y_1 = 0$$

$$\therefore \; y(x) = 1 + y_2 x^2 + y_3 x^3 + y_4 x^4 + \cdots$$

$$\frac{dy}{dx} = 2y_2 x + 3y_3 x^2 + 4y_4 x^3 + \cdots$$

$$\frac{d^2y}{dx^2} = 2y_2 + 6y_3 x + 12 y_4 x^2 + \cdots$$

Then putting this taylor expansion into eq $*$ gives:

$$2y_2 + 6y_3 x + 12y_4 x^2 + \cdots$$

$$+ \frac{2}{x}\left[2y_2 x + 3y_3 x^2 + 4y_4 x^3 + \cdots\right]$$

$$+ \left(1 + y_2 x^2 + y_3 x^3 + y_4 x^4 + \cdots\right)^n = 0$$

$$= 2y_2 + 6y_3 x + 12 y_4 x^2 + O(x^3) + 4y_2 + 6y_3 x + 8y_4 x^2 + O(x^3)$$

... continuing in this manner (using Mathematica we get)

$$= (1 + 6y_2) + 12y_3 x + (ny_2 + 20y_4)x^2 +$$

$$(ny_3 + 30y_5)x^3 + (\tfrac{1}{2}(-1+n)ny_2^2 + ny_4 + 42y_6)x^4 +$$

$$(\qquad\qquad)x^5 + (\qquad\qquad)x^6 + O(x^7) = 0$$

Setting coefficients equal to zero gives $y_2 = \frac{-1}{6} = -\frac{1}{3!}$

$$y_3 = 0$$

$$-\frac{n}{6} + 20y_4 = 0$$

$$\Rightarrow y_4 = \frac{n}{120} = \frac{n}{5!}$$

$$y_5 = 0$$

$$\tfrac{1}{2}(-1+n)ny_2^2 + ny_4 + 42y_6 = 0$$

$$\tfrac{1}{2}(-1+n)n\left(\tfrac{1}{36}\right) + n\left(\tfrac{1}{120}\right) + 42(y_6) = 0$$

$$y_6 = \frac{n(2-5n)}{3\cdot 7!}$$

$\therefore$

$$y(x) = 1 - \frac{x^2}{3!} + \frac{n}{5!}x^4 + \frac{n(2-5n)}{3\cdot 7!}x^6 + O(x^7)$$

... a slightly different $O(x^6)$ term.

$$\tilde{y}(.1) \cong 1 - \frac{(.1)^2}{3!} + n\frac{(.1)^4}{5!} + O(x^6)$$

$$\tilde{y}'(.1) = -\frac{2(.1)}{3!} + \frac{4n(.1)^3}{5!} + O(x^5)$$

$$|y(.1) - \tilde{y}(.1)| \le \left.\frac{(5n-8n^2)x^{10}}{3\cdot 7!}\right|_{\substack{x=.1\\ n=3}} = -3.76\cdot 10^{-9}$$

$$|y'(.1) - \tilde{y}'(.1)| \le \left.\frac{(5n-8n^2)6x^5}{3\cdot 7!}\right|_{\substack{x=.1\\ n=3}} = -2.26\cdot 10^{-7}$$

let $y_1(t) = y(t)$

$y_2(t) = y'(t)$

Then $\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{2}{x}y_2 - y_1^n \end{pmatrix}$

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(2.38) $y(y'')^2 = e^{2x}$

$$y(x) = x^p(a + bx + cx^2) = ax^p + bx^{p+1} + cx^{p+2}$$

$$y'(x) = px^{p-1}(a + bx + cx^2) + x^p(b + 2cx)$$

$$= apx^{p-1} + bpx^p + cpx^{p+1}$$

$$= apx^{p-1} + b(p+1)x^p + c(p+2)x^{p+1}$$

$$y''(x) = ap(p-1)x^{p-2} + b(p+1)px^{p-1} + c(p+1)(p+2)x^p$$

So $y''(x)^2 = a^2p^2(p-1)^2x^{2p-4} + 2abp(p-1)(p+1)x^{2p-3} + 2acp(p-1)(p+1)(p+2)x^{2p-2}$

$$+ b^2(p+1)^2p^2x^{2p-2} + 2bc(p+1)p(p+2)x^{2p-1} + c^2(p+1)^2(p+2)^2x^{2p}$$

$$y(y'')^2 = a^3p^2(p-1)^2x^{3p-4} + 2a^2bp(p-1)(p+1)x^{3p-3} + \cancel{} + 2a^2c(p)(p-1)(p+1)(p+2)x^{3p-2}$$

$$+ a^2p^2b(p-1)^2x^{3p-3} + 2ab^2p(p-1)(p+1)x^{3p-2}$$

$$+ \text{H.O.T.}$$

Then since $y(y'')^2 \triangleq 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!}$

We get $a^3p^2(p-1)^2 = 1 \Rightarrow a^3\left(\frac{4}{3}\right)^2\left(\frac{1}{3}\right)^2 = 1 \Rightarrow a^3 = \left(\frac{16}{81}\right)^{-1}$

$$+ \quad 3p - 4 = 0 \Rightarrow p = \frac{4}{3} \qquad \Rightarrow a = \frac{3(3)^{1/3}}{2(2)^{1/3}}$$

Since these expressions are deterministic there can only be <u>one</u> solution by this method

Given $b = \frac{27}{40} a^{-2} = \cdots$

$$y'' = \frac{\pm e^x}{\sqrt{y'}} \qquad y(0) = 0 ; \; y'(0) = 0$$

For y' increasing $\rightarrow$ $y'' > 0$ + we go to solve

$$y'' = \frac{+e^x}{\sqrt{y}} \qquad y(x_0) \cong a x_0^p + b x_0^{p+1}$$

$$+ \; y'(x_0) \cong a p x_0^{p-1} + b(p+1) x_0^p \qquad \text{For } x_0 = .001$$

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(3.1)

$$\frac{d}{dx}\left((1-x^7)\frac{dy}{dx}\right)+\lambda x^7 y=0 \quad \text{w/ B.C.}$$

$y(0)=0$ + $y'(0)$ finite

(λ is unknown)

Because this is an eigenvalue problem, an additional condition must be provided, $y(1)=1$. Requiring that $y'(1)$ bounded $\Rightarrow$ $y(1)=0$

by the arguments in the text $y'(1)=\frac{\lambda}{7}$ is the limiting behavior of $y'(x)$ at $x=1$.

Another way to look at this is to consider $y\sim 1+c(x-1)$

$$\frac{dy}{dx}\sim c$$

Then $\quad Ly\sim \frac{d}{dx}\left((1-x^7)c\right)+\lambda x^7(1+c(x-1)) = 0 \qquad x\to 1$

$$= c(-7x^6)+\lambda x^7+c\lambda x^8-c\lambda x^7$$

$$= \cancel{\lambda(1-c)x^7} \quad -7cx^6+\lambda(1-c)x^7+c\lambda x^8$$

$$= x^6\left[-7c+\lambda(1-c)x+c\lambda x^2\right] \qquad \text{now taylor expand about } x=1.$$

$$\sim \left[-7c+\lambda(1-c)+c\lambda+(\lambda(1-c)+2c\lambda)(x-1)+\cdots\right]$$

$$\sim -7c+\lambda-\cancel{\lambda c}+\cancel{c\lambda}+(\lambda(1-c)+2c\lambda)(x-1)+O((x-1)^2)$$

$\Rightarrow \quad c=\frac{\lambda}{7}$ to make the $O(1)$ term vanish

Thus $y(x) \sim 1 + \frac{d}{7}(x-1)$ as $x \to 1$

d $\therefore$ $V \sim (1-x^7)\,y' = (1-x^7)\left(\frac{d}{7}\right)$ as $x \to 1$

I7 $y_{guess}(x) = x\sin(2.5\pi x)$

$y_g' = \sin(2.5\pi x) + 2.5\pi x\cos(2.5\pi x)$

So $V_g(x) = (1-x^7)\,y_g'(x)$

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3.2

$$* \quad 2xy'' + y' = y^3 \qquad y(0) = \frac{1}{10} \qquad y(16) = \frac{1}{6}$$

Look for a solution $y \sim \alpha + px^\beta + rx$ w/ $0 < \beta < 1$

to match $y(0) = \frac{1}{10} \Rightarrow \alpha = \frac{1}{10}$. Then put $y \sim \frac{1}{10} + px^\beta + rx$ into eq $*$

+ evaluate: For this we need $y'(x) = p\beta x^{\beta-1} + r$

$$y''(x) = p\beta(\beta-1)x^{\beta-2}$$

~~$2xy''$~~

Then $2xy'' + y' - y^3 \cong 2xp\beta(\beta-1)x^{\beta-2} + p\beta x^{\beta-1} + r - \left(\frac{1}{10} + px^\beta + rx\right)^3$

$$= 2xp\beta(\beta-1)x^{\beta-2} + p\beta x^{\beta-1} + r - \left[\frac{1}{10^3} + 3\left(\frac{1}{10^2}\right)(px^\beta + rx) + \frac{3}{10}(px^\beta + rx)^2 + (px^\beta + rx)^3\right]$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ $\qquad \cong 0 \qquad x \to 0$

so

$$\Rightarrow \cancel{2p\beta(\beta-1)x^{\beta-1}} + p\beta x^{\beta-1}$$

$$= (2p\beta(\beta-1) + p\beta)x^{\beta-1} + r - \left[\frac{1}{10^3} + \frac{3}{10^2}(px^\beta + rx) + \frac{3}{10}(px^\beta + rx)^2 + (\quad)^3\right]$$

$$\Rightarrow p\beta(2\beta - 2 + 1) = 0 \Rightarrow p = 0 \text{ or } \beta = 0 \text{ or } \beta = \tfrac{1}{2}$$

Now $p=0$ is not possible or else there is not a x^{β} term.

$\beta=0$ is not possible or we have duplication of our constant term.

$\therefore\ \beta=\frac{1}{2}$ is the solution.

When $\beta=\frac{1}{2}$ the above eq becomes

$$r-\frac{1}{10^3}-\frac{3}{10^2}(px^{\beta}+rx)-\frac{3}{10}(px^{\beta}+rx)^2+(\quad)^3 \hat{=} 0$$

$$\Rightarrow\ r=\frac{1}{10^3}$$

Thus $\quad y(x)\sim \frac{1}{10}+px^{1/2}+\frac{1}{10^3}x \qquad x\to 0$ where p can be arbitrary

then $\quad y'(x)\sim \frac{1}{2}px^{-1/2}+\frac{1}{10^3} \qquad x\to 0$

Thus on $(0,d]$ we assume the above asymptotic form is true & solve the following

B.V.P.

$$y(d)=\frac{1}{10}+pd^{1/2}+\frac{1}{10^3}d \qquad y(16)=\frac{1}{6}$$

$$y'(d)=\frac{1}{2}p\left(\frac{1}{10}\right)^{-1/2}+\frac{1}{10^3}$$

Since we have introduced an unknown parameter p, we must supply a second ~~bc~~ boundary condition (that for $y'(d)$)

ode's are $\quad \frac{d}{dt}\begin{bmatrix} y_1\\ y_2\end{bmatrix}=\frac{d}{dt}\begin{bmatrix} y(t)\\ y'(t)\end{bmatrix}=\begin{bmatrix} y_2\\ \frac{y_1^3-y_2}{2x}\end{bmatrix}$

$\underline{\text{ag 145 Shampine}}$

(3.3) $y'' + \frac{2y'}{x} = \frac{y}{\varepsilon(y+k)}$

+ B.C. $y(1) = 1$ & $y'(0) = 0$

to solve this BVP on $[0,1]$ we recognize the singular nature of this differential equation on $[0,1]$ & ~~##~~ expecting a smooth solution on $[0,1)$ we taylor expand y. Assuming $y(x) \cong y_0 + y_1 x + \frac{y_2}{2!}x^2 + \frac{y_3}{3!}x^3 + O(x^4)$

$y'(0) = 0 \Rightarrow y_1 = 0$ then putting $y(x)$ into * gives

$$y_2 + y_3 x + O(y^2) + \frac{2}{x}\left(y_2 x + \frac{y_3 x^2}{2} + \cdots\right) - \frac{\left(y_0 + \frac{y_2}{2}x^2 + \frac{y_3}{6}x^3\right)}{(.1)\left(y_0 + \frac{y_2}{2}x^2 + \frac{y_3}{6}x^3 + \cdots + .1\right)} \cong 0$$

$$= y_2 + y_3 x + O(y^2) + 2y_2 + y_3 x - 10\left[\frac{y_0}{(y_0 + .1)} + \frac{d}{dx}\left(\downarrow\right)\Big|_{x=0} x + O(x^2)\right] \cong 0$$

$$\frac{y_2 x + O(x^2)}{y_0 + \frac{y_2}{2}x^2 + \frac{y_3}{6}x^3 + .1} - \frac{(y_0)(y_2 x \cdots)}{\left(y_0 + \frac{y_2}{2}x^2 + \cdots + .1\right)^2}\Bigg|_{x=0} = 0$$

$$\Rightarrow y_2 + 2y_2 - 10\left[\frac{y_0}{y_0 + .1}\right] \cong 0 \Rightarrow 3y_2 = \frac{10 y_0}{.1 + y_0}$$

$$y_2 = \frac{10}{3}\frac{y_0}{.1 + y_0}$$

$\therefore$ $y(x) \cong y_0 + \frac{10}{3}\left(\frac{y_0}{.1 + y_0}\right)x^2 + O(x^3)$ y_0 is an unknown parameter.

Thus we need to introduce a 2nd boundary condition to form a complete system, i.e. take $\cancel{y'(0)} = y(0) = Y_0 = p$ (as text explains)

3.5

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This can be shown by plugging into the DE

$$1^3 - 2(1)^2 - 1 + 2 \stackrel{?}{=} 0 \quad \checkmark$$

$$2^3 - 2(2)^2 - 2 + 2 \stackrel{?}{=} 0 \quad \checkmark$$

$$\cancel{8} - \cancel{8}$$

$$(-1)^3 - 2(-1)^2 - (-1) + 2 \stackrel{?}{=} 0$$

$$-1 - 2 + 1 + 2 = 0 \quad \checkmark$$

$y(x) = Ae^{x} + Be^{2x} + Ce^{-x}$

here

$y'(x) = Ae^{x} + 2Be^{2x} - Ce^{-x}$

- with B.C. $y(0) = 1$, $y(+\infty) = 0$, $y'(+\infty) = 0$ would require

$$A + B + C = 1$$

$$A \cdot \infty + B \cdot \infty + 0 = 0$$

$$A \cdot \infty + 2B \cdot \infty + 0 = 0$$

$\Rightarrow$ $A = -B$, then $-B + 2B = 0 \Rightarrow B = 0$, then $A = 0$ & $C = 1$

$\therefore \quad y(x) = e^{-x}$.

- w/ B.C. $y(0) = 1$, $y'(0) = 1$, $y(+\infty) = 0$

$$A + B + C = 1 \quad (1)$$

$$A + 2B - C = 1 \quad (2)$$

$$A \cdot \infty + B \cdot \infty = 0 \Rightarrow A = -B \text{ then in eq (1) + (2)}$$

$-B + B + C = 1 \Rightarrow C = 1$ into (2) $B - 1 = 1 \Rightarrow B = 2$

& $-B + 2B - C = 1$ then $A = -2$

$\therefore \quad y(x) = -2e^{x} + 2e^{2x} + e^{-x}$

$y(+\infty) = -2e^{+\infty} + 2e^{2\infty} \neq 0$

$y'(x) = -2e^{x} + 4e^{2x} - e^{-x}$

$y'(0) = -2+4-1 = 1$ ✓

looks like this would work but since $e^{2x} \gg e^{x} \iff e^{x} \gg 1 \quad x \to \infty$

$e^{x} = \Omega(e^{2x}) \quad \therefore \quad y(+\infty) = +\infty$.

to show that this is true: consider

$y(0)=1, \; y'(0)=1, \; y(20)=0$

$\to \quad A+B+C=1$

$A+2B-C=1$

$Ae^{20}+Be^{40}+Ce^{-20} = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ e^{20} & e^{40} & e^{-20} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Check the condition # of this matrix:
Using Matlab we get:
$\text{cond}(A) \cong 1.6 \cdot 10^{17}$!! huge

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & e^{40}-e^{20} & e^{-20}-e^{20} & e^{20} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & \cancel{e^{-20}-e^{40}} & e^{20} \end{array}\right]$$

$2e^{40}-3e^{20}+e^{-20}$

$\cancel{-e^{40}+e^{20}+e^{-20}-e^{20} = -e^{40}+e^{-20}}$

$-2(-e^{40}+e^{20})+e^{-20}-e^{20}$

$2e^{40}-2e^{20}+e^{-20}-e^{20}$

$2e^{40}-3e^{20}+e^{-20}$

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(3.6)

$$f^{iv} + (\Omega + f)f''' + \Omega f f'' = \gamma\left[f'f'' + \Omega(f')^2\right] \quad *$$

w/ B.C. $f(0) = 0 \quad f'(0) = 0$

& as $x \to +\infty$

$$f'(x) \sim e^{-\Omega x}, \quad f''(x) \sim -\Omega e^{-\Omega x}$$

consider solutions to * w/ $f(x) \sim \delta + \rho e^{-\lambda x}$ ✓

$$f'(x) \sim -\lambda\rho e^{-\lambda x} \quad ✓$$

$$f''(x) \sim \lambda^2\rho e^{-\lambda x} \quad ✓$$

$$f'''(x) \sim -\lambda^3\rho e^{-\lambda x} \quad ✓$$

$$f^{iv}(x) \sim \lambda^4\rho e^{-\lambda x} \quad ✓$$

$$\Rightarrow \lambda^4\rho e^{-\lambda x} + (\Omega + \delta + \rho e^{-\lambda x})(-\lambda^3\rho e^{-\lambda x}) + \Omega(\delta + \rho e^{-\lambda x})(\lambda^2\rho e^{-\lambda x}) \quad ✓$$

$$= \gamma\left[-\lambda\rho e^{-\lambda x}(\lambda^2\rho e^{-\lambda x}) + \Omega\lambda^2\rho^2 e^{-2\lambda x}\right] \quad ✓$$

$\div \lambda^2\rho e^{-\lambda x}$ to get

$$\lambda^2 + (\Omega + \delta + \rho e^{-\lambda x})(-\lambda) + \Omega(\delta + \rho e^{-\lambda x}) = \quad ✓$$

$$= \gamma\left[-\lambda\rho e^{-\lambda x} + \Omega\rho e^{-\lambda x}\right] \quad ✓$$

$$\Rightarrow \quad \lambda^2 - \lambda(\Omega+\delta) + \Omega\delta + (-\lambda\rho + \Omega\rho + \gamma\lambda\rho - \gamma\Omega\rho)e^{-\lambda x} = 0$$

$\{\delta, \rho, \lambda \text{ unknown}\}$ set each term to zero.

$$\lambda^2 - \lambda(\Omega+\delta) + \Omega\delta = 0 \quad *$$

$$\& \quad -\lambda\rho + \Omega\rho + \gamma\lambda\rho - \gamma\Omega\rho = 0 \quad **$$

2 eqs & 3 unknowns hence an underdetermined system, need another eq.

From the B.C. at $x \to +\infty$ $f' \sim e^{-\Omega x}$ $\Rightarrow$ choose to $\lambda = \Omega$ then in $*$

$$\Omega^2 - \Omega^2 - \Omega\delta + \Omega\delta = 0 \quad ✓$$

$$-\cancel{\Omega}\rho + \cancel{\Omega}\rho + \gamma\cancel{\Omega}\rho - \gamma\cancel{\Omega}\rho = 0$$

$\therefore$ ρ & δ are underdetermined, then to match B.C. at $+\infty$ pick

$$\rho = -\frac{1}{\Omega} \quad \text{so} \quad f(x) \sim \delta - \frac{1}{\Omega}e^{-\Omega x}$$

From this there exist an <u>exact</u> solution to the O.D.E. $\Rightarrow$ $f(0) = 0$

is satisfied $\quad f(0) = \delta - \frac{1}{\Omega} = 0 \Rightarrow \delta = \frac{1}{\Omega}$

so $f(x) = \frac{1}{\Omega}(1 - e^{-\Omega x})$; $f'(x) = e^{-\Omega x}$; $f''(x) = -\Omega e^{-\Omega x}$

$$f^{IV}(x) = \Omega^2 e^{-\Omega x}$$

let

$$\underline{y} = \begin{pmatrix} f \\ f' \\ f'' \\ f''' \end{pmatrix} \quad \text{Then} \quad \frac{d}{dx}\underline{y} = \begin{pmatrix} f' \\ f'' \\ f''' \\ (-\Omega+f)f''' + \gamma\left[f'f'' + \Omega(f')^2\right] \end{pmatrix}$$

$$\text{so} \quad \frac{d}{dx}\underline{y} = \begin{bmatrix} y_2(x) \\ y_3(x) \\ y_4(x) \\ (-\Omega + y_1(x))y_4(x) + \gamma\left[y_2(x)y_3(x) + \Omega(y_2(x))^2\right] \end{bmatrix}$$

w/ B.C. $\quad y_1(0) = 0 \ , \ y_2(0) = 0, \ y_2(\bar{x}) = e^{-\Omega\bar{x}}, \ y_3(\bar{x}) = -\Omega e^{-\Omega x}$

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$f'' = \beta g f \quad g'' = -\beta g h \quad h'' = \lambda \beta g h$

$$\frac{g''}{h''} = \frac{-\beta}{\lambda\beta} = -\frac{1}{\lambda}$$

$$\Rightarrow g'' = -\frac{1}{\lambda} h''$$

$$\Rightarrow g'(z) = -\frac{1}{\lambda} h'(z) + c_1$$

$$\Rightarrow g(z) = -\frac{h(z)}{\lambda} + c_1 z + c_2$$

Since $z = x - t$ when $t \to \pm\infty$ g, h shall go to constants

$\therefore c_1 = 0$ let $c_2 = E$

$$\therefore g(z) = E - \frac{h(z)}{\lambda}$$

So $h'' = \lambda\beta\left(E - \frac{h(z)}{\lambda}\right)h$ + $f'' = \beta\left(E - \frac{h(z)}{\lambda}\right)f$

w/ B.C. $h(0) = 1$, $f(0) = 1$, $h(\infty) = 0$, $f(\infty) = 0$

w/ $\beta = 10$, $\lambda = 10$, + $E = 1$

the above eqs become

$$h'' = 100\left(1 - \frac{h}{10}\right)h \quad + \quad f'' = 10\left(1 - \frac{h}{10}\right)f$$

$$\Rightarrow h'' - 100h = -10h^2$$

to derive an asymptotic form for $h(z)$ as $z \to \infty$

$$\cancel{h''(z) - 100h(z)} \quad {\sim \cancel{0 \; z \to +\infty}}$$

$$h''(z) - 100h(z) = -10\cancel{k^2(z)} \quad \sim 0 \quad z \to +\infty$$

$$\therefore \quad h(z) \cong C_1 e^{10z} + C_2 e^{-10z} \qquad z \to +\infty$$

$$\therefore \quad h(z) \sim e^{-10z} \qquad z \to +\infty$$

Then $\quad k''(z) - 10k(z) = -\cancel{h(z)}k(z) \quad \sim 0 \quad z \to +\infty$

$$\therefore \quad k(z) \sim C_1 e^{-\sqrt{10}\,z} + C_2 e^{+\sqrt{10}\,z} \qquad \text{since } k(+\infty) = 0 \quad C_2 = 0$$

$$\& \quad k(z) \sim e^{-\sqrt{10}\,z} \qquad z \to +\infty$$

Then the B.C. at $z = +\infty$ can be replaced by B.C. at some finite boundary z. (This is because the exponential decay is so strong that k & h are so close to zero for moderate to large z. that ~~the~~ the error in taking the B.C. to be identical to zero should be negligible.)

The 1st order system to solve is

$$\underline{y} = \begin{pmatrix} k(z) \\ k'(z) \\ h(z) \\ h'(z) \end{pmatrix} \qquad \text{Then} \quad \frac{dy}{dz} = \begin{pmatrix} k'(z) \\ \lambda \cdot \beta\left(E - \frac{h(z)}{\lambda}\right)h \\ h'(z) \\ \beta\left(E - \frac{h}{\lambda}\right)k \end{pmatrix} = \begin{pmatrix} y_2(z) \\ \lambda \cdot \beta\left(E - \frac{y_3(z)}{\lambda}\right)y_3(z) \\ y_4(z) \\ \beta\left(E - \frac{y_3(z)}{\lambda}\right)y_1(z) \end{pmatrix}$$

3.8 $$w''(z) + \frac{2z}{\sqrt{1-\alpha w(z)}}\, w'(z) = 0$$

w/ BC $w(0)=1 \qquad w(+\infty) = 0 \qquad 0 \le \alpha \le 1$

$\alpha = .8$

The ODE becomes approx

$$w''(z) + 2z\, w'(z) = 0 \qquad z \to +\infty$$

$$\frac{w''(z)}{w'(z)} = -2z$$

$$\ln w'(z) = -z^2 + C_1$$

$$w'(z) = C_2 e^{-z^2}$$

$$\Rightarrow \quad w(z) = C_2 \int_0^z e^{-\xi^2}\, d\xi + C_3$$

$$w(+\infty) = 0 \Rightarrow \quad C_2 \underbrace{\int_0^\infty e^{-\xi^2}\, d\xi}_{\frac{\sqrt{\pi}}{2}} + C_3 = 0$$

$$\therefore \quad C_3 = -C_2 \frac{\sqrt{\pi}}{2}$$

so $$w(z) = C_2 \int_0^z e^{-\xi^2} d\xi - C_2 \frac{\sqrt{\pi}}{2} = C_2 \underbrace{\left(\int_0^z - \int_0^\infty \right)}_{} e^{-\xi^2} d\xi$$

$$-C_2 \int_z^\infty e^{-\xi^2}\, d\xi$$

since $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$ & $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt$

$$w(z) = -C_2\frac{\sqrt{\pi}}{2}\operatorname{erfc}(z)$$

By standard asymptotic analysis $\operatorname{erfc}(z) \sim \frac{e^{-z^2}}{z\sqrt{\pi}}$
see (Bender & Orszag)

$$w(z) \sim -C_2\frac{\sqrt{\pi}}{2}\frac{e^{-z^2}}{z\sqrt{\pi}}$$

that C_2 is an unknown & enforce another B.C

$$w'(z) = C_2 e^{-z^2} \qquad w(z) = \left(-C_2\frac{\sqrt{\pi}}{2}\right)\operatorname{erfc}(z)$$

$z = z_c$ end of interval

(3.9)

$$y'' = x^{-1/2} y^{3/2} \iff (y'')^2 = x^{-1} y^3$$

$$y(0) = 1 \qquad y(+\infty) = 0$$

$$y(0) = 1 \Rightarrow y'' \sim x^{-1/2} \qquad x \to 0$$

$$y' \sim \frac{x^{1/2}}{1/2} = 2x^{1/2}$$

$$y \sim \frac{2x^{3/2}}{3/2} = \frac{4}{3}x^{3/2}$$ but w/ just this term $y(0) \neq 1$

$$y(x) = 1 + px + \frac{4}{3}x^{3/2} + bx^2 + cx^{5/2} + \dots$$

then $$y'(x) = p + 2x^{1/2} + 2bx + \frac{5}{2}cx^{3/2} + \dots$$

$$y''(x) = x^{-1/2} + 2b + \frac{15}{4}cx^{1/2} + \dots$$

Then $$(y''(x))^2 = x^{-1} + 4bx^{-1/2} + \frac{15}{2}c(1) + O(x^{1/2})$$
$$+ 4b^2 + \frac{4b}{4}\cdot 15cx^{1/2} + O(x^1)$$
$$+ O(x^1)$$

$$(y(x))^3 = \left(1 + px + \frac{4}{3}x^{3/2} + bx^2 + cx^{5/2} + \dots\right)^3$$
$$= \left(1 + px + \frac{4}{3}x^{3/2} + bx^2 + cx^{5/2} + \dots\right)\left(1 + px + \frac{4}{3}x^{3/2} + bx^2 + cx^{5/2} + \dots\right)^2$$
$$= \left(1 + px + \frac{4}{3}x^{3/2} + bx^2 + cx^{5/2} + \dots\right)\left(1 + 2px + \frac{8}{3}x^{3/2} + 2bx^2 + 2cx^{5/2} + \dots + p^2x^2 + \frac{8}{3}px^{5/2} + \dots\right)$$

$$= \quad 1 + 2px + \frac{8}{3}x^{3/2} + 2bx^2 + 2cx^{5/2} + \cdots$$
$$+ px \qquad\qquad + 2p^2x^2 + \frac{8}{3}px^{5/2} + \cdots$$
$$+ \frac{4}{3}x^{3/2} + \qquad + \frac{8}{3}px^{5/2} + \cdots$$

$$= 1 + 3px + 4x^{3/2} + 3bx^2 + 3p^2x^2 + O(x^{5/2})$$ checking w/ MMA

So $\quad x^{-1}y^3 = x^{-1} + 3p + 4x^{1/2} + 3bx + 3p^2x + O(x^{3/2})$

$$+ \quad (y'')^2 - x^{-1}y^3 = \frac{4b}{\sqrt{x}} + \left(4b^2 + \frac{15c}{2} - 3p\right) + (-4 + 15bc)\sqrt{x}$$
$$+ \left(-3b + \frac{225c^2}{16} - 3p^2\right)x + \cdots$$

setting as many terms to zero as we can we get

$b = 0 \qquad \frac{15c}{2} - 3p = 0 \quad \Rightarrow \quad c = \frac{6}{15}p = \frac{2}{5}p$ w/ p undetermined

at this stage

— — —

$\therefore \; y(x) = 1 + px + \frac{4}{3}x^{3/2} + \frac{2}{5}px^{5/2} \qquad x \to 0$

— — —

As $x \to +\infty$

Assume $y(x) \sim a x^{\alpha}$

$$y'(x) \sim a\alpha x^{\alpha-1}$$

$$y''(x) = a\alpha(\alpha-1) x^{\alpha-2}$$

So $x(y'')^2 - y^3 = x a^2 \alpha^2 (\alpha-1)^2 x^{2\alpha-4} - a^3 x^{3\alpha} =$

$$a^2 \alpha^2 (\alpha-1)^2 x^{2\alpha-3} - a^3 x^{3\alpha}$$

require $\quad 2\alpha - 3 = 3\alpha \Rightarrow \alpha = -3$

& then $\quad a^2\left[\alpha^2(\alpha-1)^2 - a\right] = 0$

$$9(16)\ \cancel{9(8)} - a = 0 \Rightarrow a = \cancel{72}\ 144.$$

$\therefore\ y(x) = 144 x^{-3}$

Now let $y(x) = 144x^{-3} + \varepsilon(x)$ w/ $\varepsilon(x) \ll 144x^{-3}$

Then $y(x)^{3/2} = \left(y_0(x) + \varepsilon(x)\right)^{3/2} = y_0(x)^{3/2}\left(1 + \dfrac{\varepsilon(x)}{y_0(x)}\right)^{3/2} \cong y_0(x)^{3/2}\left(1 + \dfrac{3}{2}\dfrac{\varepsilon}{y_0}\right)$

$y(x)^{3/2} = y_0(x)^{3/2} + \dfrac{3}{2} y_0(x)^{1/2} \varepsilon(x)$ then in

$$y'' = x^{-1/2} y^{3/2}$$

$$\cancel{y_0''} + \varepsilon'' = x^{-1/2}\left[\cancel{y_0^{3/2}} + \frac{3}{2} y_0^{1/2} \varepsilon\right]$$

$\Rightarrow \quad \varepsilon'' = \frac{3}{2} x^{-1/2} y_0^{1/2} \varepsilon \qquad w/ \quad y_0 = 144 x^{-3}$

$\Rightarrow \quad \varepsilon'' = \frac{3}{2} x^{-1/2} \, 12 x^{-3/2} \varepsilon \; = 18 x^{-2} \varepsilon$

+ consider $\quad \varepsilon(x) = c x^{r}$

$\varepsilon' = c r x^{r-1}$

$\varepsilon'' = c r (r-1) x^{r-2}$ put in the above

$c r (r-1) \cancel{x^{r-2}} = 18 \cancel{x^{-2}} c \cancel{x^{r}}$

$\Rightarrow \quad r(r-1) = 18 \quad \Rightarrow \quad r^2 - r - 18 = 0$

$r = \frac{1}{2}(1 - \sqrt{73}) \;,\; \frac{1}{2}(1 + \sqrt{73})$

$\doteq -3.772 \quad + \quad 4.772$

Since $\quad \varepsilon(x) \ll y_0(x) = O(x^{-3}) \quad$ as $\quad x \to +\infty$

only the 1st root is possible

$\therefore \quad y(x) \doteq 144 x^{-3} + c x^{r} \qquad w/ \quad r = \frac{1}{2}(1 - \sqrt{73})$

So ODE is $\quad y'' = x^{-1/2} y^{3/2}$

+ BC are $\quad y(d) = 1 + p x + \frac{4}{3} x^{3/2} + \frac{2p}{5} x^{5/2} \Big|_{x=d}$

$y'(d) = p + 2 x^{1/2} + p x^{3/2} \Big|_{x=d}$

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$$+ \quad y(\bar{x}) = 144\,\bar{x}^{-3} + c(\bar{x})^{r} \qquad r = \frac{(1-\sqrt{73})}{2}$$

$$y'(\bar{x}) = -3(144)\bar{x}^{-4} + cr(\bar{x})^{r-1}$$

3.10 $y'' + 100y = 0$

$y(0) = 1$ & $y(10) = B$

$y'' + 10^2 y = 0$ has solution $y(x) = A\cos 10x + B\sin(10x)$

so $y(0) = 1$

is $A\cos(0) + B\cdot 0 = 1 \Rightarrow A = 1$

&

$$\begin{bmatrix} \cos(10\cdot 0) & \sin(10\cdot 0) \\ \cos(10\cdot 10) & \sin(10\cdot 10) \end{bmatrix} \begin{bmatrix} A_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ B \end{bmatrix}$$

$$\| $$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ .8623 & -.5064 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

has condition # of 3.6778 ... No problem inverting.

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(3.11) From the discussion in the text

$$\delta_i = \frac{2}{h_i}\tau_i \quad \text{w/ } \tau_i \text{ the truncation error which for the}$$

trapezoidal rule is given by

$$\tau_i = y(x_{i+1}) - y_i - \frac{h_i}{2}\left[f(x_i,y_i) + f(x_{i+1},y_{i+1})\right]$$

$$= \cancel{y(x_i)} + \cancel{f(x_i,y_i)h_i} + \frac{d^2y}{dx^2}\cdot\frac{h_i^2}{2} + \frac{d^3y}{dx^3}\frac{h_i^3}{6} + O(h_i^4)$$

$$-\cancel{y_i}$$

$$-\frac{h_i}{2}\cancel{f(x_i,y_i)}$$

$$-\frac{h_i}{2}\Big[\cancel{f(x_i,y_i)} + f_x(\cdot,\cdot)h_i + f_y(\cdot,\cdot)\Delta y + \frac{f_{xx}(\cdot,\cdot)h_i^2}{2} + f_{xy}(\cdot,\cdot)h_i\Delta y + f_{yy}(\cdot,\cdot)\frac{\Delta y^2}{2}$$

$$+ O(h^3)\Big]$$

- - - -

Now $\frac{dy}{dx} = f(x,y)$

$$\frac{d^2y}{dx^2} = f_x + f_y y_x = f_x + f f_y$$

$$\frac{d^3y}{dx^3} = f_{xx} + f_{xy}f + f_x f_y + f_y f\cdot f_y + f(f_{xy} + f_{yy}f)$$

$$= f_{xx} + 2f f_{xy} + f_x f_y + f f_y^2 + f^2 f_{yy}$$

So

$$\tau_i = \left(\cancel{f_x(\cdot,\cdot)} + f(\cdot,\cdot)f_y(\cdot,\cdot)\right)\frac{h_i^2}{2} + \frac{d^3y}{dx^3}\frac{h_i^3}{6} + O(h_i^4)$$

$$-\frac{h_i}{2}\left[f_x(\cdot,\cdot)h_i + f_y(\cdot,\cdot)\Delta y + f_{xx}(\cdot,\cdot)\frac{h_i^2}{2} + f_{xy}(\cdot,\cdot)h_i\Delta y + f_{yy}(\cdot,\cdot)\frac{\Delta y^2}{2} + O(h_i^3)\right]$$

$$\Delta y = y_{i+1} - y_i = f(x_i,y_i)h_i + \frac{d^2y}{dx^2}\frac{h_i^2}{2} + O(h_i^3)$$

$$\Rightarrow \tau_i = \cancel{f\cdot f_x\frac{h_i^2}{2}} + \frac{d^3y}{dx^3}\frac{h_i^3}{6} + O(h^4)$$

$$-\frac{h_i}{2}\left[\cancel{f_y f h} + f_y\frac{d^2y}{dx^2}\frac{h^2}{2} + O(h^3) + f_{xx}\frac{h^2}{2} + f_{xy}h\Delta y + f_{yy}\frac{h^2}{2} + O(h^3)\right]$$

$$= \frac{d^3y}{dx^3}\frac{h_i^3}{6} - \frac{h^3}{4}f_y\frac{d^2y}{dx^2} = O(h^3)$$

$\sigma_i = k h^p y^{(p+1)}(\eta_i)$ $\qquad p = 3 \quad k = ?$ $\qquad$ Not see what is wrong...

p

(3,12)

consider:

$$y_{i+1/2} = y_i + h_i\left[\frac{5}{24} f(x_i,y_i) + \frac{1}{3} f(x_{i+1/2}, y_{i+1/2}) + -\frac{1}{24} f(x_{i+1},y_{i+1})\right] \quad (1)$$

$$y_{i+1} = y_i + h_i\left[\frac{1}{6} f(x_i,y_i) + \frac{2}{3} f(x_{i+1/2}, y_{i+1/2}) + \frac{1}{6} f(x_{i+1},y_{i+1})\right] \quad (2)$$

$$\Rightarrow \quad y_{i+1/2} - \frac{h_i}{3} f(x_{i+1/2}, y_{i+1/2}) = y_i + h_i\left[\frac{5}{24} f(x_i,y_i) - \frac{1}{24} f(x_{i+1},y_{i+1})\right]$$

Use eq (2) solving for $f(x_{i+1/2}, y_{i+1/2})$ gives

$$\left(\frac{y_{i+1} - y_i}{h_i} - \frac{1}{6} f(x_i,y_i) - \frac{1}{6} f(x_{i+1},y_{i+1})\right) \frac{3}{2} = f(x_{i+1/2}, y_{i+1/2})$$

So

$$y_{i+1/2} = y_i + h_i\left[\frac{5}{24} f(x_i,y_i) + \frac{1}{3}\cdot\frac{3}{2}\left(\frac{y_{i+1} - y_i}{h_i} - \frac{1}{6} f(x_i,y_i) - \frac{1}{6} f(x_{i+1},y_{i+1})\right) - \frac{1}{24} f(x_{i+1},y_{i+1})\right]$$

$$= y_i + h_i\left[\frac{1}{2}\frac{(y_{i+1} - y_i)}{h_i} + \left(\frac{5}{24} - \frac{1}{12}\right) f(x_i,y_i) + \left(-\frac{1}{12} - \frac{1}{24}\right) f(x_{i+1},y_{i+1})\right]$$

$$= y_i + \frac{y_{i+1}}{2} - \frac{y_i}{2} + h_i\left[\underset{\overset{\parallel}{\frac{1}{8}}}{\frac{3}{24}} f(x_i,y_i) - \frac{1}{8} f(x_{i+1},y_{i+1})\right]$$

$$= \cancel{\frac{1}{2}(y_i + y_{i+1})}$$

$$= \frac{1}{2}(y_i + y_{i+1}) + \frac{h_i}{8}\left[f(x_i,y_i) - f(x_{i+1},y_{i+1})\right]$$

then put this back in eq (2)

So

$$y_{i+1} = y_i + h_i\left[\frac{1}{6}f(x_i, y_i) + \frac{2}{3}f\left(x_{i+1/2},\ \frac{1}{2}(y_i + y_{i+1}) + \frac{h_i}{8}\big(f(x_i, y_i) - f(x_{i+1}, y_{i+1})\big)\right) + \frac{1}{6}f(x_{i+1}, y_{i+1})\right]$$

(3.13) If the ODE is $y' = J(x)y + q(x)$

$$B_a y(a) + B_b y(b) = 0$$

So the system one gets when using the condensed simpson rule is

$$f(x, y) = J(x)y + q(x)$$

So

$$\cancel{f(x_{i+1/2}, \ldots)}$$

$$f(x_i, y_i) - f(x_{i+1}, y_{i+1}) = J(x_i)\,y_i + q(x_i) - J(x_{i+1})\,y_{i+1} - q(x_{i+1})$$

$$= J(x_i)\,y_i - J(x_{i+1})\,y_{i+1} + q(x_i) - q(x_{i+1})$$

So

$$\frac{1}{2}(y_i + y_{i+1}) + \frac{h_i}{8}\big(f(x_i, y_i) - f(x_{i+1}, y_{i+1})\big) = \left(\frac{I}{2} + \frac{J(x_i)h}{8}\right)y_i$$

$$+ \left(\frac{I}{2} - \frac{J(x_{i+1})h}{8}\right)y_{i+1} + \frac{h}{8}q_i - \frac{h}{8}q_{i+1}$$

So

$$\Downarrow$$

$$f(x_{i+1/2}, \ldots) = J(x_{i+1/2})\left[\left(\frac{I}{2} + \frac{J(x_i)h}{8}\right)y_i + \left(\frac{I}{2} - \frac{J(x_{i+1})h}{8}\right)y_{i+1} + \frac{h}{8}q_i - \frac{h}{8}q_{i+1}\right]$$

$$+ q(x_{i+1/2})$$

$$= \left[\frac{J_{i+1/2}}{2} + \frac{J_{i+1/2} J_i h}{8}\right] y_i + \left[\frac{J_{i+1/2}}{2} + \frac{J_{i+1/2} \cdot J_{i+1} h}{8}\right] y_{i+1}$$

$$+ \frac{h}{8} J_{i+1/2} q_i - \frac{h}{8} J_{i+1/2} q_{i+1} + q_{i+1/2}$$

Then

$$y_{i+1} = y_i + \frac{h}{6}\left[J(x_i) y_i + q_i\right] + \frac{2h}{3}\left[\frac{J_{i+1/2}}{2} + \frac{J_{i+1/2} J_i h}{8}\right] y_i + \frac{2h}{3}\left[\frac{J_{i+1/2}}{2} + \frac{J_{i+1/2} J_{i+1} h}{8}\right] y_{i+1}$$

$$+ \frac{2}{3}\frac{h^2}{8} J_{i+1/2} q_i - \frac{2}{3}\frac{h^2}{8} J_{i+1/2} q_{i+1} + \frac{2h}{3} q_{i+1/2}$$

$$+ \frac{h}{6}\left[J_{i+1} y_{i+1} + q_{i+1}\right]$$

$$\Rightarrow \underbrace{\left[I + \frac{h}{6} J_i + \frac{h}{3} J_{i+1/2} + \frac{J_{i+1/2} J_i h}{12}\right]}_{S_i} y_i + \underbrace{\left[-I + \frac{h}{6} J_{i+1} + \frac{h}{3} J_{i+1/2} + \frac{J_{i+1/2} J_{i+1} h}{12}\right]}_{R_i} y_{i+1}$$

$$= \underbrace{-\frac{h}{6} q_i - \frac{h}{6} q_{i+1} - \frac{1}{12} h^2 J_{i+1/2} q_i + \frac{1}{12} h^2 J_{i+1/2} q_{i+1}}_{V_i} - \frac{2h}{3} q_{i+1/2} \qquad *.$$

$$+ \text{ B.C} \qquad B_a y_a + B_b y_b = 0$$

$$\Rightarrow \begin{bmatrix} S_0 & R_0 & & & \\ & S_1 & R_0 & & \\ & & \ddots & & \\ & & & S_{N-1} & R_{N-1} \\ B_a & & & & B_b \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} V_0 \\ \\ \vdots \\ V_{N-1} \\ \beta \end{bmatrix}$$

$S(x)$ is a cubic polynomial on $[x_i, x_{i+1}]$

$\Rightarrow$ 4 coefficients

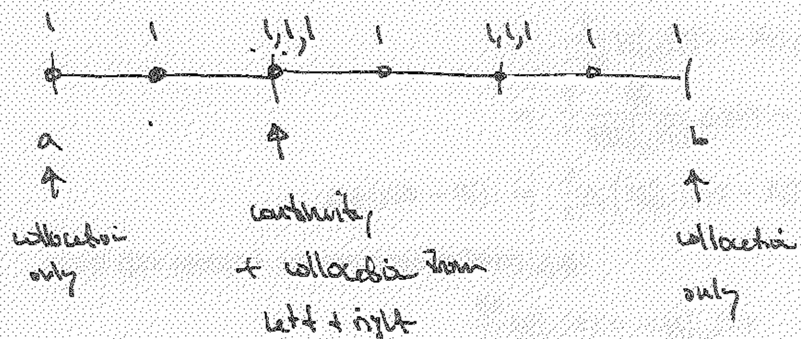
+ require $S'(x_i) = f(x_i, S(x_i))$ + $S(x)$ is continuous on $[a,b]$

$S'(x_{i+1/2}) = f(x_{i+1/2}, S(x_{i+1/2}))$

$S'(x_{i+1}) = f(x_{i+1}, S(x_{i+1}))$

$\Rightarrow$ $S_L(x_i) = S_R(x_i)$, sim for x_{i+1}.

count the # of eqs we have:



$= 11$ eqs + B.C. $g(S(a), S(b)) = 0$

$=$ ~~10 eqs~~ 12 eqs ✓

total # of unknowns $= 3 \text{ cells} \cdot 4 = 12$ ✓

Since $S(\cdot)$ are forced to be continuous & by collocation

$S'(x_i) = f(x_i, S(x_i))$ the derivative of S is continuous

$\Rightarrow$ ~~$S(\cdot)$~~ $S(\cdot) \in C^1([a,b])$

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$$y(x) = ax^{\alpha} + bx^{\beta} + \cdots$$
$$z(x) = 1 + cx^{\gamma} + dx^{\delta} + \cdots$$

$$0 = 3yy'' - 2(y' - z) \qquad * \qquad \begin{array}{l} 0 < \alpha < \beta \\ 0 < \gamma < \delta \end{array}$$

So $$y'(x) = a\alpha x^{\alpha-1} + b\beta x^{\beta-1} + \cdots$$
$$y''(x) = a\alpha(\alpha-1)x^{\alpha-2} + b\beta(\beta-1)x^{\beta-2} + \cdots$$

$$z'(x) = c\gamma x^{\gamma-1} + d\delta x^{\delta-1}$$
$$z''(x) = c\gamma(\gamma-1)x^{\gamma-2} + d\delta(\delta-1)x^{\delta-2}$$ into * give

$$0 = 3\left[ax^{\alpha} + bx^{\beta} + \cdots\right]\left[a\alpha(\alpha-1)x^{\alpha-2} + b\beta(\beta-1)x^{\beta-2} + \cdots\right]$$
$$-2\left[a\alpha x^{\alpha-1} + b\beta x^{\beta-1} - 1 - cx^{\gamma} - dx^{\delta} + \cdots\right]$$

$$= 3\left[a^2\alpha(\alpha-1)x^{2\alpha-2} + ab\beta(\beta-1)x^{\alpha+\beta-2} + \cdots\right.$$
$$\left. + ab\alpha(\alpha-1)x^{\alpha+\beta-2} + b^2\beta(\beta-1)x^{2\beta-2} + \cdots\right]$$
$$-2\left[a\alpha x^{\alpha-1}\right.$$

3.15

$$y'' + \lambda e^{y} = 0$$

$$y(0) = 0 = y(1)$$

Multiply by $2y'(x)$ + integrate

$$\Rightarrow \frac{d}{dx}\left(y'(x)^2\right) + 2\lambda \frac{d}{dx} e^{y(x)} = 0$$

$$\Rightarrow y'(x)^2 + 2\lambda e^{y(x)} = C_1.$$

evaluating at $x=0$ gives

$$y'(0)^2 + 2\lambda e^{0} = C_1 \Rightarrow C_1 = 2\lambda + y'(0)^2$$

$\therefore$

$$y'(x)^2 + 2\lambda e^{y(x)} = 2\lambda + y'(0)^2$$

$$\Rightarrow y'(x)^2 + 2\lambda\left(e^{y(x)} - 1\right) = y'(0)^2 \quad \text{conservation law.}$$

(3.16)

$$y' = \tan(\phi)$$

$$v' = -\frac{(g\sin(\phi) + \nu v^2)}{v\cos(\phi)}$$

$$\phi' = -\frac{g}{v^2}$$

w/ $g = .032$ + $\nu = .02$ on $t \in [0, 5]$

$v(0) = .5$

$y(0) = 0 = y(5)$

⋮ see matlab file prob_3_16.m

(3.17) $y'' = P_e(y' + Ry^n)$

+ B.C. $y'(0) = P_e(y(0) - 1)$ $y'(1) = 0$

$P_e = 1$, $R = 2$, $n = 2$

see matlab file prob_3_17.m

(3.18) $$\frac{d^2\phi}{ds^2} + \frac{P}{B}\cos(\phi) = 0$$

w/ BC $\phi(0) = 0$ $\phi'(L) = 0$

since $\frac{dx}{ds} = \cos(\phi)$ + $\frac{dy}{ds} = -\sin(\phi)$

$+ \quad x(0) = 0 \,, \quad y(0) = 1$

$L = 10 \quad + \quad \frac{P}{B} = 0.001$

$\vdots$

(3.19) $$y'' + \omega^2\left(\frac{1-\alpha^2}{H}\frac{1}{\sqrt{1+\varepsilon^2 y^2}} + \alpha^2\right)y = 0$$

$+$ B.C $y'(0) = 0 \,,\; y'(1) = 0$

$0 \le \alpha < 1$ but ω is unknown $\Rightarrow$ we need another B.C to determine ω. consider $y(0) = 1$.

Now H is defined via

$$H = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)\int_0^1 \frac{dx}{\sqrt{1+\varepsilon^2 y^2(x)}}\right]$$

consider $\cancel{h(x) = \frac{1}{\alpha^2}\left[1-(1-\alpha^2)\int_0^x \frac{dx}{\sqrt{1+\varepsilon^2 y^2(x)}}\right]}$

~~then~~ $\cancel{\frac{dh}{dx} = -\frac{(1-\alpha^2)}{\alpha^2}\frac{1}{\sqrt{1+\varepsilon^2 y^2(x)}}}$

$$y_3(x) = \int_0^x \frac{d\xi}{\sqrt{1+\varepsilon^2 y^2(\xi)}} \qquad \frac{dy_3}{dx} = \frac{1}{\sqrt{1+\varepsilon^2 y^2(x)}} \;; \quad y_3(0) = 0$$

$$+ \quad y_4(x) = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)\int_0^1 \frac{dx}{\sqrt{\quad}}\right] = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)\,y_3(1)\right]$$

+ $\frac{dy_4}{dx} = 0$ w/ B.C. $y_4(1) = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)y_3(1)\right]$

Note we could have done $y_4(0) = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)y_3(1)\right]$ but then the B.C. would not be separated. + separated B.C. are easier to deal w/

Thus the system + B.C. are (w/ $y_1(x) = y(x)$, $y_2(x) = y'(x)$)

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_2(x) \\ -\omega^2\left(\frac{1-\alpha^2}{y_4(x)}\frac{1}{\sqrt{1+\varepsilon^2 y_1(x)^2}} + \alpha^2\right)y_1(x) \\ \frac{1}{\sqrt{1+\varepsilon^2 y_1(x)^2}} \\ 0 \end{pmatrix}$$

+ B.C.'s:

$y_2(0) = 0$; $y_2(1) = 0$; $y_1(0) = 1$ (for ω determination) ; $y_3(0) = 0$;

$y_4(1) = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)y_3(1)\right]$

Now consider $\varepsilon = 0$ the DE becomes

$$y'' + \omega^2\left(\frac{1-\alpha^2}{H} + \alpha^2\right)y = 0$$

+ $H = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)\int_0^1 1 \cdot dx\right] = \frac{1}{\alpha^2}\left[1 - 1 + \alpha^2\right] = 1$

So ODE becomes

$$y'' + \omega^2(1 - x^2 + x^2)y = 0$$

$$\Rightarrow \quad y'' + \omega^2 y = 0 \qquad \text{\& B.C. } \begin{array}{l} y'(0) = 0 \\ y'(1) = 0 \end{array}$$

$$\Rightarrow \quad y = A\cos(\omega x) + B\sin(\omega x)$$

$$y'(x) = -A\omega\sin(\omega x) + \omega B\cos(\omega x)$$

$$y'(0) = \omega B = 0 \quad \Rightarrow \quad B = 0.$$

$$y'(1) = -A\omega\sin(\omega x) = 0 \quad \Rightarrow \quad \omega = n\pi \qquad n = \pm 1, \pm 2, \ldots$$

only positive values of n correspond to linearly independent solutions.

$$\Rightarrow \quad y(x) = A\cos(n\pi x)$$

to have $y(0) = 1 \Rightarrow A = 1$

$$y(x) = \cos(n\pi x)$$

Certainly $n = 1$ is a solution $y(x) = \cos(\pi x)$

3,20

$$y' = D_a(1-y)\exp\left(\frac{\gamma\theta}{\gamma+\theta}\right)$$

$$\theta' = BD_a(1-y)\exp\left(\frac{\gamma\theta}{\gamma+\theta}\right) - \beta(\theta-\theta_c)$$

$y(0) = (1-\lambda)y(1) \qquad \theta(0) = (1-\lambda)\theta(1)$

⋮

3.21

$$y''(t) = \frac{y'(t)(1+(y'(t))^2)}{t(3(y'(t))^2-1)}$$

+ $\quad y(t_0)=0,\ y'(t_0)=1,\ y(1)=1 \qquad t_0 \in [0,1]$ is unknown

+ $k = t_0^2 + \displaystyle\int_{t_0}^{1} \frac{2\tau}{(y'(\tau))^2+1}\,d\tau$

let $\quad y_1(t) = y(t)$
$\quad\ \ y_2(t) = y'(t)$

+ let $\quad y_3(t) = t^2 + \displaystyle\int_{t}^{1} \frac{2\tau}{y'(\tau)^2+1}\,d\tau \qquad y_3(t_0) = k$ desired

+ $y_3(1) = 1 + 0$

+ So that

$$y_3'(t) = 2t - \frac{2t}{y'(t)^2+1} = 2t\left[\frac{y'(t)^2+1-1}{y'(t)^2+1}\right] = \frac{2t\ y'(t)^2}{y'(t)^2+1} \quad ✓$$

We now need to change to a fixed interval.

let $x(t) = \dfrac{t - t_0}{1 - t_0}$ $\quad \left\{ \underbrace{x(t_0) = 0}_{} \ \& \ x(1) = 1 \right.$ ✓

w/ $\dfrac{dx}{dt} = \dfrac{1}{1-t_0}$ & $t = (1-t_0)x + t_0$ ✓

So $\dfrac{dF}{dt} = \dfrac{dx}{dt}\dfrac{dF}{dx} = \left(\dfrac{1}{1-t_0}\right)\dfrac{dF}{dx}$

$\therefore$ ODE system is w/ $y_1 = y$; ~~...~~ $y_2 = y'$; $y_3 = t^2 + \displaystyle\int_t^1 \frac{2\tau\, d\tau}{(y'(\tau))^2 + 1}$

$$\frac{1}{(1-t_0)}\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_2 \\ \dfrac{y_2(1+y_2^2)}{[t_0 + (1-t_0)x](3y_2^2 - 1)} \\ \dfrac{2[t_0 + (1-t_0)x]\, y_2^2}{y_2^2 + 1} \end{pmatrix} \checkmark$$

w/ $y_1(0) = 0$; $y_2(0) = 1$; $y_1(1) = 1$; $y_3(1) = 1$

or $$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} (1-t_0)y_2 \\ \dfrac{1-t_0}{(t_0 + (1-t_0)x)} \dfrac{y_2(1+y_2^2)}{(3y_2^2 - 1)} \\ \dfrac{2(1-t_0)(t_0 + (1-t_0)x)\, y_2^2}{(y_2^2 + 1)} \end{pmatrix}$$ w/ B.C. from above. ✓

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(3.22) $\theta'' + \sin(\theta) = 0$

$\theta(0) = 0 \;;\; \theta(+\infty) = \pi$

pick $T \gg 1$ & solve w/ B.C. $\theta(T) = \pi$.

check that $\theta'(0) = 2$.

⋮

(3.23) $f''' + ff'' + \beta(1 - (f')^2) = 0$

subject to $f(0) = 0$; $f'(0) = 0$; $f'(+\infty) = 1$

w/ $-.19.. \le \beta \le 2$ pick $X \ni f'(X) = 1$

⋮

(3.24) $G' - xF' = 0$ $G(0) = F(0) = 0$

$F'' + 2(xF - G)F' + 2(1 - F^2) = 0$ $F(+\infty) = 1$

$F'(0) = ?$

Assume $G(x)$ has a limit as $x \to +\infty$. integrate eq * at $x \to +\infty$

$G(x) - \int xF' = C$

$\Rightarrow G(x) - xF + \int F = C$ $x \to +\infty$

*

$G' - \frac{d}{dx}(xF) + F = 0$

$G(x) - xF + \int F = C$

As $x \to \infty$ $G(x)$ goes to $G(x)$ $x \gg 1$

xF goes to x since $F \sim 1$ $x \gg 1$

$\int F$

I don't see how to get the belief that $xF - G \sim x$?

But assuming it is correct in the 2nd eq we can get an eq for F.

$$F'' + 2xF' \doteq 0 \qquad x \to +\infty$$

$$\frac{F''}{F'} = -2x$$

$$\ln F' = -x^2 + C \Rightarrow F'(x) = \beta e^{-x^2}$$

$$F(x) = \beta \int_0^x e^{-\xi^2} d\xi + C \quad \text{w/ } F(+\infty) = 1 \Rightarrow \beta\left(\frac{\sqrt{\pi}}{2}\right) + C = 1$$

since $\cancel{\int_0^x e^{-t^2} dt = \ldots}$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

& $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$

~~we can write~~

So

$$F(x) = \beta \int_0^x e^{-\xi^2} d\xi + 1 - \beta\left(\frac{\sqrt{\pi}}{2}\right)$$

$$= 1 + \beta \underbrace{\left[\int_0^x e^{-\xi^2} d\xi - \underset{\displaystyle \int_0^\infty}{\overset{\displaystyle \frac{\sqrt{\pi}}{2}}{\|}} \right]}_{\displaystyle -\int_x^\infty e^{-\xi^2} d\xi}$$

$\Rightarrow \cancel{F(x) = 1 - \beta \ldots \sqrt{\pi}}$

$$\Rightarrow \quad F(x) = 1 - \beta \sqrt{\frac{\pi}{2}}\, \mathrm{erfc}(x) \quad \checkmark$$

$+$ w/ $\mathrm{erfc}(x) \sim \dfrac{e^{-x^2}}{x\sqrt{\pi}}$

$$G' = xF' = x\beta e^{-x^2}$$

$$G(x) = \frac{\beta}{2} e^{-x^2} + G(\infty)$$

$\therefore$ since both F + G have ~~~~ ~~~~ exponential decay to their limiting values one cannot hope to compute on very large intervals.

3.26 $\xi = z/c \quad q(\xi) = U(z)$

$$c^{-2} q'' + q' + q(1-q) = 0 \qquad q(-\infty) = 1 \quad q(\infty) = 0$$

$c \gg 1$ solve $q_0' + q_0(1-q_0) = 0$

$$\Rightarrow \quad \frac{dq}{d\xi} = -q(1-q)$$

$$\frac{dq}{q(1-q)} = -d\xi$$

$$\Rightarrow \quad \frac{1}{q} + \frac{1}{1-q} = -d\xi$$

$$\ln q - \ln(1-q) = -\xi + C_1$$

$$\frac{q}{1-q} = C_2 e^{-\xi}$$

$$q = C_2 e^{-\xi} - q e^{-\xi} C_2$$

$$(1 + C_2 e^{-\xi}) q = C_2 e^{-\xi} \quad \Rightarrow \quad q = \frac{C_2 e^{-\xi}}{1 + C_2 e^{-\xi}} = \frac{1}{\frac{1}{C_2} e^{\xi} + 1}$$

$q(+\infty) = 0$ ✓

$q(-\infty) = 1$ ✓ $\qquad C_2$ undetermined. take $= 1$

$\therefore$ outer solution is $U(z) \cong \dfrac{1}{1 + e^{z/c}}$ & $U'(z) = \cancel{\dfrac{(-1)(1/c)}{(1+e^{z/c})^2}}$

$$= \frac{(-1)(1/c)\, e^{z/c}}{(1 + e^{z/c})^2}$$

To show that this is a good approximation to the true travelling ~~wa~~ wave solution to Fisher's Eq

As $z \to -\infty$

$$U(z) \approx (1+e^{z/c})^{-1} \cong 1 - e^{+z/c}$$

As $z \to +\infty$

$$U(z) = \frac{e^{-z/c}}{(1+e^{-z/c})} = e^{-z/c}(1+e^{-z/c})^{-1} \cong e^{-z/c}(1-e^{-z/c}) \cong e^{-z/c}$$

So to match the true behavior $U(z) \sim e^{\beta z}$ w/ $\beta = \frac{-c+\sqrt{c^2-4}}{2}$

we see that in the choice $\beta' = -\frac{1}{c}$ when the exact value of (As $z \to +\infty$)

$$\beta \text{ is } = -\frac{c}{2} + \frac{1}{2}(c^2-4)^{1/2} = -\frac{c}{2} + \frac{c}{2}(1-4/c^2)^{1/2}$$

$$= \cancel{-\frac{c}{2}} + \frac{c}{2}(\cancel{1} - \frac{2}{c^2}) = -\frac{1}{c} \quad \checkmark \text{ the same.}$$

+ As $z \to -\infty$ $U'(z) \sim \alpha(U(z)-1)$ where our approximate solution satisfies

$$U'(z) \cong -\frac{1}{c}e^{z/c} = -\frac{1}{c}(1-U) = \frac{1}{c}(U-1) \quad \text{so} \quad \alpha' = \frac{1}{c}$$

The true solution has $\alpha = -\frac{c}{2} + \frac{\sqrt{c^2+4}}{2} = -\frac{c}{2} + \frac{c}{2}(1+\frac{4}{c^2})^{1/2}$

$$\cong -\frac{c}{2} + \frac{c}{2}(1+\frac{2}{c^2}) = \frac{1}{c} \quad \checkmark \text{ the same.}$$

3,30

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \quad + \quad \underline{C} = \begin{bmatrix} Y_1(1) \\ Y_2(1) \\ Y_3(1) \end{bmatrix}$$

so $\underline{Y}(1) = \underline{Y}(0)$ can be expressed in terms of $\underline{C}$ as

$$\underline{C}(0) = \underline{Y}(0) \quad + \quad C(1) = Y(1)$$

M 216 Sherpin

Ex 4.1 $\quad y'(t) = (1+y(t))\, y(t-1)$

$1 \le t \le 3$ w/ history $y = 1 \quad 0 \le t \le 1$

on $\quad 1 \le t \le 2$

$y'(t) = (1+y(t))\, y(t-1) = 1 + y(t) \qquad$ since $\; y(t-1) = 1$

$\Rightarrow \quad y'(t) - y(t) = 1 \quad \to \quad \frac{d}{dt}\left(e^{-t} y(t)\right) = e^{-t}$

$\Rightarrow \quad e^{-t} y(t) = -e^{-t} + C$

$y(t) = Ce^{t} - 1 \quad$ w/ $\; y(1) = 1 \Rightarrow Ce^{1} - 1 = 1$

$C = 2e^{-1}$

$\therefore \; y(t) = 2e^{t-1} - 1 \qquad 1 \le t \le 2$

Now on $\quad 2 \le t \le 3$

$y'(t) = (1 + y(t))(2e^{t-1} - 1) \quad$ w/ $\quad y(2) = 2e^{1} - 1$

Solving w/ Mathematica gives:

$y(t) = -1 + C e^{2e^{t-1}}$

$y(2) = \cancel{-1} + Ce^{2e} = 2e - \cancel{1}$

$C = 2e \cdot e^{-2e} = 2e^{1-2e}$

$\therefore \; y(t) =$

- $y'(t) = (1 + y(t))\,y(t/2) \qquad 1 \le t \le 4$

  w/ $y(t) = 1 \qquad \frac{1}{2} \le t \le 1$

For ~~$0 \le t \le 1$ the ODE is~~ $1 \le t \le 2$ the ODE is

$$y'(t) = (1+y(t))(\cdot 1) = 1 + y(t) \qquad \text{since } \tfrac{1}{2} \le \tfrac{t}{2} \le 1 \text{ \& there } y \equiv 1$$

$$\Rightarrow y(t) = Ce^{t} + t \qquad + \quad y(1) = 1$$

$$\Rightarrow Ce + 1 = 1 \Rightarrow C = 0 \quad \therefore y(t) = t$$

Now on $2 \le t \le 4$ $1 \le \frac{t}{2} \le 2$ so the DE is

$$y'(t) = (1 + y(t)) \cdot t \qquad \text{w/ } y(2) = 2$$

$$\Rightarrow y'(t) - t\,y(t) = t$$

Solving w/ Mathematica gives:

$y(t) =$

I expect discontinuities to appear at $1, 2, 4, 8, 16, 32 \ldots 2^n$.

- $y'(t) = (1+y^2(t))\,y(t-1)$

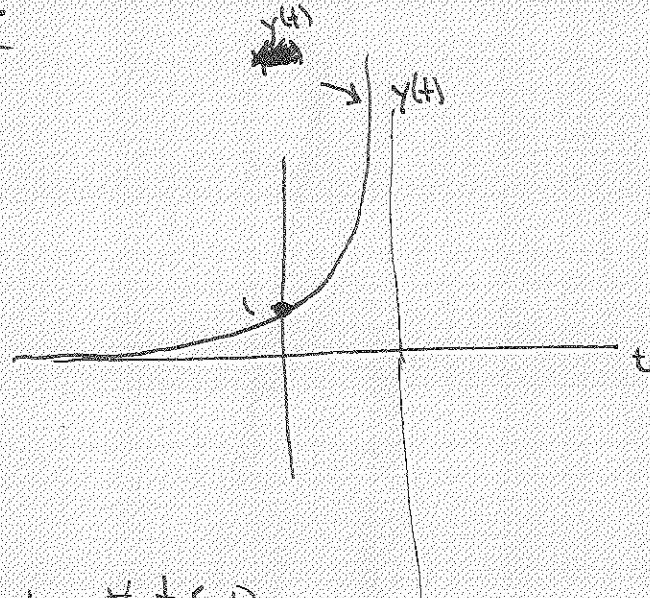
Then for $1 \le t \le 2$ we have $0 \le t-1 \le 1$ so the ODE to

solve is $y'(t) = 1+y^2(t)$. & $\cancel{y(2)}$ $y(1)=1$ we have

∘ $y'(t) = y^2(t) \qquad y(0) = 1$

$\Rightarrow \dfrac{dy}{y^2} = dt$

$-\dfrac{1}{y} = t + c \Rightarrow y = \dfrac{-1}{t+c}$



So $1 = \dfrac{-1}{c} \Rightarrow c = -1$

$\therefore \; y(t) = \dfrac{-1}{t-1} = \dfrac{1}{1-t}$

Thus y is not defined for $t \geq 1$

v.s. $y'(t) = y(t)\,y(t-\tau)$ w/ $y(t) = 1 \;\; \forall\, t < 0$

Let $y_k(t)$ be the solution to DDE on $[k\tau, (k+1)\tau]$.

Now $y_0(t)$ is the solution to

$y'(t) = y(t) \qquad t \in [0, \tau] \qquad y(0) = 1$

$\Rightarrow y(t) = e^t$ which exists & is continuous.

Assume $y_k(t)$ exists & is continuous

Then $y_{k+1}(t)$ is the solution to

$y_{k+1}'(t) = y_{k+1}(t)\, y_k(t) \qquad y_{k+1}((k+1)\tau) = y_k(\tau k)$

Then $y_{k+1}(t)$ exists + is continuous by theory of ODE's.

$\{ F(y_{k+1})$ is linear + certainly Lipschitz $\}$ this solution exists

$\forall\ t \in [(k+1)\tau, (k+2)\tau]$. By induction ~~for~~ a solution to the DDE

exists $\forall$ time

- $y' = -y \Rightarrow y = Ce^{-t}$

$y'(t) = -y(t - \frac{\pi}{2})$ has $y = A\sin(t) + B\cos(t)$ as solutions

$$\sin(t - \tfrac{\pi}{2}) = \sin(t)\cancel{\cos(\tfrac{\pi}{2})}^{0} - 1\cos t$$

$y' = A\cos t - B\sin t$

$$\cos(t - \tfrac{\pi}{2}) = \cos(\ )\cancel{\cos(\ )}^{0} + \sin(t)$$

$+\ y(t - \frac{\pi}{2}) = \cancel{A}\cos t$

$= A(-1)\cos t + B\sin t$

So $y' = -y(t - \frac{\pi}{2})$ very different solutions

pg 227 Shampine

$I_1(t) =$

$$I_2(t) = \int_{t-D}^{t} \int_{s}^{t} e^{\sigma w} x(w) d(w)\, dw\, ds$$

$$= \int_{t-D}^{t} F(t,s)\, ds \quad \text{w/} \quad F(t,s) = \int_{s}^{t} e^{\sigma w} x(w) d(w)\, dw$$

$$\text{So} \quad I_2'(t) = F(t,t) - F(t,t-D) + \int_{t-D}^{t} F_t(t,s)\, ds$$

$$= 0 - \int_{t-D}^{t} e^{\sigma w} x(w) d(w)\, dw + \int_{t-D}^{t} e^{\sigma t} x(t) d(t)\, ds$$

$$= \int_{t-D}^{t} e^{\sigma t} x(t) d(t)\, ds - \int_{t-D}^{t} e^{\sigma w} x(w) d(w)\, dw$$

eq on bottom of pg 227. ✓

Mg 239 Shenjian

(4.5) Example 4.4.3 is
when $t \le D$

$$x'(t) = -x(t)\lambda(t) + by(t)$$

$$n = x(t) + y(t)$$

$$\lambda(t) = \frac{c}{n}\left\{\int_0^t \int_s^t e^{-b(t-w)} x(w)\lambda(w)\,dw\,ds + \int_0^t e^{-b(t-s)} y(s)\,ds\right\}$$

+ when $t \ge D$

$$x'(t) = -x(t)\lambda(t) + by(t)$$

$$n = x(t) + y(t)$$

$$\lambda(t) = \frac{c}{n}\left\{\int_{t-D}^t \int_s^t e^{-b(t-w)} x(w)\lambda(w)\,dw\,ds + \int_{t-D}^t e^{-b(t-s)} y(s)\,ds\right\}$$

Following the hint in the book assume x, y, λ are constant as $t \to +\infty$

Then we get the following system of equations:

$$0 = -x\lambda + by$$

$$n = x + y$$

$$\lambda = \frac{c}{n}\left\{\int_{t-D}^t \int_s^t e^{-b(t-w)} x\lambda\,dw\,ds + y e^{-bt}\int_{t-D}^t e^{bs}\,ds\right\}$$

Eq. for λ becomes:

$$\lambda = \frac{c}{n}\left\{ x\lambda e^{-\sigma t}\int_{t-\Delta}^{t}\int_{s}^{t} e^{\sigma w}\,dw\,ds + y e^{-\sigma t}\frac{e^{\sigma s}}{\sigma}\Big|_{t-\Delta}^{t} \right\}$$

$$= \frac{c}{n}\left\{ x\lambda e^{-\sigma t}\int_{t-\Delta}^{t}\frac{e^{\sigma w}}{\sigma}\Big|_{s}^{t} ds + \frac{y e^{-\sigma t}}{\sigma}\left(e^{\sigma t} - e^{\sigma(t-\Delta)}\right) \right\}$$

$$= \frac{c}{n}\left\{ \frac{x\lambda}{\sigma} e^{-\sigma t}\int_{t-\Delta}^{t}\left(e^{\sigma t} - e^{\sigma s}\right) ds + \quad '' \right.$$

$$= \frac{c}{n}\left\{ \frac{x\lambda}{\sigma}\left[t-(t-\Delta) - e^{-\sigma t}\frac{e^{\sigma w}}{\sigma}\Big|_{t-\Delta}^{t} \right] + \frac{y}{\sigma}\left(1 - e^{-\sigma\Delta}\right) \right\}$$

$$= \frac{c}{n}\left\{ \frac{x\lambda}{\sigma}\left[\Delta - \frac{e^{-\sigma t}}{\sigma}\left(e^{\sigma t} - e^{\sigma(t-\Delta)}\right) \right] + \frac{y}{\sigma}\left(1 - e^{-\sigma\Delta}\right) \right\}$$

$$= \frac{c}{n}\left\{ \frac{x\lambda}{\sigma}\left[\Delta - \frac{1}{\sigma}\left(1 - e^{-\sigma\Delta}\right) \right] + \frac{y}{\sigma}\left(1 - e^{-\sigma\Delta}\right) \right\}$$

$$= \frac{c}{n\sigma}\left\{ x\lambda\Delta - \frac{x\lambda}{\sigma} + \frac{x\lambda}{\sigma}e^{-\sigma\Delta} + y - y e^{-\sigma\Delta} \right\}$$

$$(-x\lambda + y\sigma)\frac{1}{\sigma}e^{-\sigma\Delta}(-1) = 0 \text{ from 1st condition}$$

$$= \frac{c}{n\sigma}\left\{ x\lambda\Delta \underbrace{- \frac{x\lambda}{\sigma} + y}_{=0 \text{ by 1st condition}} \right\}$$

$\therefore$ we have the following system: (x, λ, y)

$$0 = -x\lambda + Gy$$

$$n = x + y$$

$$\lambda = \frac{C}{nG} x \lambda D \Rightarrow \lambda = 0 \text{ or } 1 = \frac{C x D}{nG} = \frac{CD}{G}\left(\frac{x}{n}\right)$$

If $\lambda = 0$ then $y = 0$ & $x = n$.

otherwise

$$0 = -x\lambda + Gy$$

$$n = x + y$$

$$x = n\left(\frac{G}{CD}\right)$$

put $x = n\left(\frac{G}{CD}\right)$ into eq (2) $\Rightarrow$ $y = \left(1 - \frac{G}{CD}\right)n$ then into eq (1)

$\Rightarrow$

$$0 = -n\left(\frac{G}{CD}\right)\lambda + Gn\left(1 - \frac{G}{CD}\right)$$

since $n \neq 0$ & $G \neq 0$

$$\Rightarrow \frac{\lambda}{CD} = 1 - \frac{G}{CD} \Rightarrow \lambda = CD - G$$

So $\lambda = CD - G$ & $x = \frac{nG}{CD} = \frac{nG}{\lambda + G}$ & $y = n - x$ ✓

Ex 4.11

The steady state solution is when $y_1'(t) = 0 = y_2'(t)$

or & $y_1(t-\tau) = y_1$

$$y_1 - \frac{y_1^3}{3} - y_2 + m\cancelto{0}{(y_1 - y_{1,0})} = 0 \qquad (1)$$

& $$r(y_1 + a - by_2) = 0 \qquad (2)$$

eq (2) gives $y_1 = by_2 - a$ $\quad r \neq 0$ put into eq (1) gives

$$by_2 - a - \frac{(by_2 - a)^3}{3} - y_2 = 0$$

$$\Rightarrow by_2 - a - \frac{1}{3}\left(b^3 y_2^3 - 3b^2 y_2^2 a + 3by_2 a^2 - a^3\right) - y_2 = 0$$

$$\Rightarrow -\frac{1}{3}b^3 y_2^3 + b^2 a y_2^2 + (b - 1 - ba^2)y_2 + \left(-a + \frac{a^3}{3}\right) = 0$$

4.13 Equilibrium pt has $x' = 0 = y'$ + No delay.

$$\Rightarrow \quad x\left[2\left(1-\frac{x}{40}\right) - \frac{y}{x+10}\right] - 10 = 0$$

$$+ \quad y\left[\frac{x}{x+10} - \frac{2}{3}\right] = 0$$

~~$x \neq 0$~~ $\exists$ $y \neq 0$ gives for the second equation

$$x = \frac{2}{3}(x+10)$$

$$\frac{1}{3}x = \frac{20}{3} \Rightarrow x = 20 \quad \text{Then the 1st eq gives}$$

$$20\left[2\left(1-\frac{1}{2}\right) - \frac{y}{30}\right] - 10 = 0$$

$$20\left[1 - \frac{y}{30}\right] = 10 \Rightarrow 1 - \frac{y}{30} = \frac{1}{2}$$

$$\Rightarrow 1 - \frac{1}{2} = \frac{y}{30} \Rightarrow y = 15$$

$\therefore$ our equilibrium pt is $(20,15)$

Again solve

$$x\left[2\left(1-\frac{x}{80}\right) - \frac{y}{x+40}\right] - 10 = 0$$

$$+ \quad y\left[-3 + \frac{6x}{x+40}\right] = 0$$

Assuming $y \neq 0$

$$\frac{2x}{x+40} = 1$$

$$2x = x + 40 \Rightarrow x = 40 \quad \text{then}$$

$$40\left[2\left(1-\frac{4}{5}\right) - \frac{y}{80}\right] = 10 \quad = 0$$

$$40\left[2\left(\frac{1}{5}\right) - \frac{y}{80}\right] - 10 = 0$$

$$\Rightarrow \frac{2}{5} - \frac{y}{80} - \frac{1}{4} = 0$$

$$\frac{2}{5} - \frac{1}{4} = \frac{y}{80} \Rightarrow \underbrace{\frac{8}{20} - \frac{5}{10}} = \frac{y}{80}$$

$$\frac{3}{20} = \frac{y}{80} \Rightarrow y = 12$$

Again $\quad x\left[20 - x - y\right] - 7 = 0$

$$+ \quad -15y + 3xy = 0$$

$$y \neq 0 \Rightarrow x = 5. \quad \text{then}$$

$$5(20 - 5 - y) - 7 = 0$$

$$15 - y = \frac{7}{5} \Rightarrow y = 15 - \frac{7}{5} = \frac{68}{5} = \checkmark$$