MJ 53/BBoss

$$\operatorname{cov}\Big(\sum_{j\neq i} x_j, x_i\Big) = \sum_{j\neq i} \operatorname{cov}(x_j, x_i) = \sum_{j\neq i} 0$$

x_j + x_i are independent.

$$\operatorname{Var}\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{1}^{n} \operatorname{Var}(X_i) + 2\sum_{i<j}\sum \operatorname{cov}(X_i, X_j)$$

$$\begin{aligned}
\operatorname{cov}(x_i, x_j) &= E[x_i x_j] - E[x_i]E[x_j] \\
&= P\{x_i = 1, x_j = 1\} - p^2 \\
&= P\{x_i = 1\} P\{x_j = 1 \mid x_i = 1\} - p^2 \\
&= \left(\frac{Np}{N}\right) \frac{(Np-1)}{N-1} - p^2
\end{aligned}$$

$=$

$$\begin{aligned}
\operatorname{Var}\Big(\sum x_i\Big) &= np(1-p) + 2\binom{n}{2}\left[p\,\frac{(Np-1)}{N-1} - p^2\right] \\
&= np(1-p) + 2\,\frac{n(n-1)}{2}\,p\left[\frac{Np-1}{N-1} - p\right] \\
&= np(1-p) + np(n-1)\left[\frac{Np-1}{N-1} - p\right]
\end{aligned}$$

$$= np(1-p) + np(n-1)\left[\frac{\cancel{N}p - 1 - p\cancel{N} + p}{N-1}\right]$$

$$= np(1-p) + \frac{np(n-1)(p-1)}{N-1} \quad \checkmark$$

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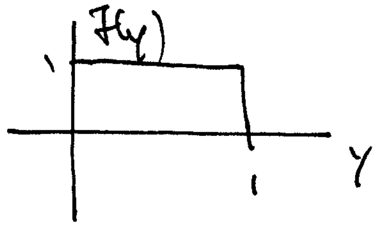
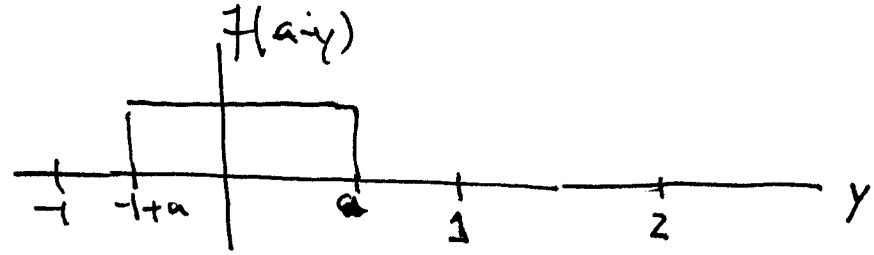
$f(a-y) = f(-(y-a))$





$a < 0 \quad f_{X+Y}(a) = 0$

$a > 2 \quad f_{X+Y}(a) = 0$

$0 \le a \le 1 \quad f_{X+Y}(a) = \int_0^a 1\,dy = a$

$1 \le a \le 2 \quad f_{X+Y}(a) = \int_{-1+a}^{1} 1\,dy = 1-(-1+a) = 1+1-a = 2-a$

$X_{(i)}$ = ith smallest RV.

$$P\{X_{(i)} \le x\} = \sum_{k=i}^{n} \binom{n}{k} F(x)^k (1-F(x))^{n-k}$$

$$f_{X_{(i)}}(x) = f(x)\sum_{k=i}^{n} k\binom{n}{k}F^{k-1}(1-F)^{n-k} - \underbrace{f(x)\sum_{k=i}^{n} (n-k)\binom{n}{k}F^{k}(1-F)^{n-k-1}}_{k=n \text{ term vanishes} \ldots}$$ ✓

$$- f(x)\sum_{k=i}^{n-1} (n-k)\binom{n}{k}F^{k}(1-F)^{n-k-1}$$ ✓

subtract 1 from k index

$k \to k-1$

=

$$= f(x)\sum_{k=i}^{n} k\binom{n}{k}F^{k-1}(1-F)^{n-k} - f(x)\sum_{k=i+1}^{n} (n-k+1)\binom{n}{k-1}F^{k-1}(1-F)^{n-k}$$ ✓

$$= \cancel{f(x)} \cancel{- f(x)(n-i-1+1)\binom{n}{i}F^{i}(1-F)^{n-i-1}}$$

$$\cancel{+ f(x)\sum_{k=i+1}^{n}\left[k\binom{n}{k} - (n-k+1)\binom{n}{k-1}\right]F^{k-1}(1-F)^{n-k}}$$

$$= f(x) \frac{n!}{(n-i)!\,(i-1)!} F^{i-1}(1-F)^{n-i}$$

$$+ f(x) \sum_{k=i+1}^{n} \left[\cancelto{0}{k\binom{n}{k} - (n-k+1)\binom{n}{k-1}} \right] F^{k-1}(1-F)^{n-k}$$

$$\cancelto{0}{\frac{n!}{(k-1)!\,(n-k)!} - \frac{n!}{(n-k)!\,(k-1)!}} \qquad\qquad n-k+1$$

$$= f(x) \frac{n!}{(n-i)!\,(i-1)!} F^{i-1}(1-F)^{n-i} \qquad \checkmark$$

\- - - - - -

For $i-1$ values to be less than x has prob. $F(x)^{i-1}$ } 3 events
$n-i$ values to be greater than x has prob $(1-F)^{n-i}$
to have value x has prob $f(x)$. By multinomial dist here

$$\frac{(n-i+(i-1)+1)!}{(n-i)!\,(i-1)!\,1!} = \frac{n!}{(n-i)!\,(i-1)!}$$ ways in which this can happen.

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① 5R
30
2B

2 Balls selected

$S = \{ (R,R), (R,O), (R,B), (O,O), (O,B), (B,B) \}$

$X \in \{0, 1, 2\}$

$P\{X = 0\} = \dfrac{3}{6} = \dfrac{1}{2}$

$P\{X = 1\} = \dfrac{2}{6} = \dfrac{1}{3}$

$P\{X = 2\} = \dfrac{1}{6} = \dfrac{1}{6}$

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(2) $X = H - T$

$X \in$ ~~$\{n, n-1, n-2, \ldots, -n+1, -n\}$~~

$\{n, n-2, n-4, \ldots, -n+2, -n\}$

(3) $n = 2$

$X = $ ~~$\{2, 1, 0, -1, -2\}$~~ $\{2, 0, -2\}$

$p(2) = \frac{1}{4} = \binom{2}{2}\left(\frac{1}{2}\right)^2$ Binomial

~~$p(1) = \binom{2}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$~~

$p(0) = \binom{2}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$

~~$p(-1) =$~~

$p(-2) = \binom{2}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

(4)

(i) $S = \{1, 2, \ldots, 6\}$

(ii) $S = \{1, 2, \ldots, 6\}$

(iii) $S = \{2, 3, \ldots, 12\}$

(iv) $\{1, 2, \ldots, 6\} - \{1, 2, \ldots, 6\} \Rightarrow S = \{0, -1, \ldots, -5,$

$1, 0, -1, \ldots, -4,$

2

$\vdots$

$5, 4, 3, 2, 1, 0\}$

$\Rightarrow S = \{-5, -4, -3, \ldots, +3, +4, +5\}$

(5)

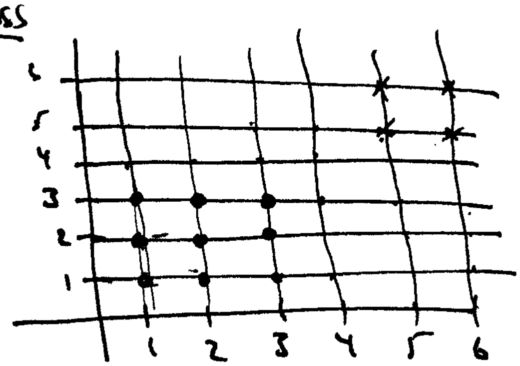
(i) $P_1 = 1\left(\frac{1}{6}\right)^2$

$P_2 = 3\left(\frac{1}{6}\right)^2$

$P_3 = 5\left(\frac{1}{6}\right)^2$

$P_4 = \frac{7}{6^2}$

$P_5 = \frac{9}{36}$; $P_6 = \frac{11}{36}$



Check $1 + 3 + 5 + 7 + 9 + 11 = 36$ ✓.

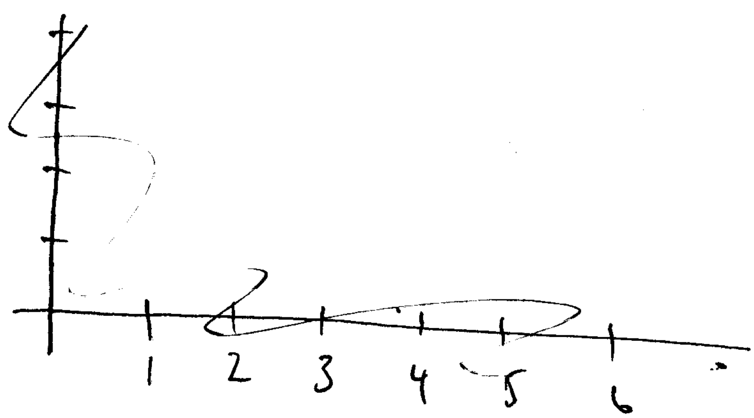
(ii) $P_6 = \frac{1}{36}$ $P_4 = \frac{5}{36}$ $P_2 = \frac{9}{36}$

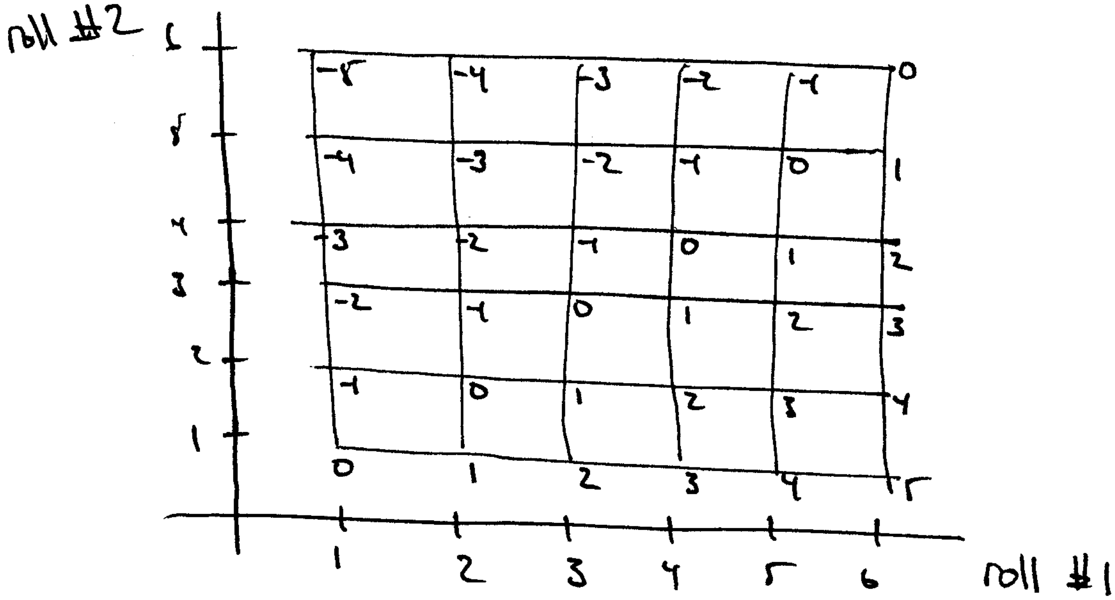
$P_5 = \frac{3}{36}$ $P_3 = \frac{7}{36}$ $P_1 = \frac{11}{36}$

(iii) $P_2 = \frac{1}{36}$; $P_3 = \frac{2}{36}$; $P_4 = \frac{3}{36}$; $P_5 = \frac{4}{36}$; $P_6 = \frac{5}{36}$

$P_7 = \frac{6}{36}$; $P_8 = \frac{5}{36}$; $P_9 = \frac{4}{36}$; $P_{10} = \frac{3}{36}$; $P_{11} = \frac{2}{36}$; $P_{12} = \frac{1}{36}$

(iv)





$P_{-5} = \frac{1}{36}$; $P_{-4} = \frac{2}{36}$; $P_{-3} = \frac{3}{36}$; $P_{-2} = \frac{4}{36}$; $P_{-1} = \frac{5}{36}$

$P_0 = \frac{6}{36}$; $P_1 = P_{-1}$; $P_2 = P_{-2}$; $P_3 = P_{-3}$; $P_4 = P_{-4}$; $P_5 = P_{-5}$.

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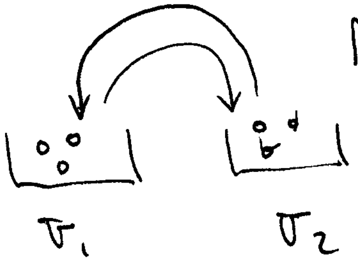
(1)



3 white
3 black

state i if 1st urn contains i white balls.

$\{X_n, n = 0, 1, 2, \dots\}$ is a markov chain iff

$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0\} =$

$P\{X_{n+1} = j \mid X_n = i\}$ which is true in this case since state at the next time step is independent of all previous states.

Require P_{ij} $\quad 0 \le i \le 3, 0 \le i \le 3$

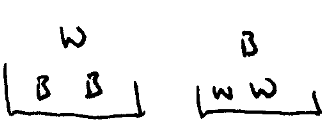
$= P\{X_{n+1} = j \mid X_n = i\}$

$$\begin{bmatrix} (0,0) & (0,1) & (0,2) & (0,3) \\ (1,0) & (1,1) & (1,2) & (1,3) \\ (2,0) & (2,1) & (2,2) & (2,3) \\ (3,0) & (3,1) & (3,2) & (3,3) \end{bmatrix}$$

$P_{00} = 0 \qquad P_{01} = 1 \qquad P_{02} = 0 \qquad P_{03} = 0$

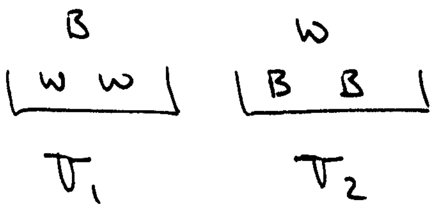
$P_{10} = (\frac{1}{3})^2 = \frac{1}{9}$; $\quad P_{11} = (\frac{1}{3})\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3}$;

$= \frac{4}{9}$

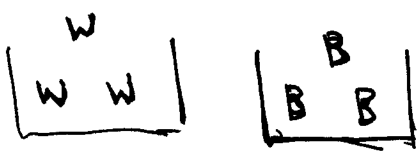


$= P_{12} = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$; $P_{13} = 0$

$P_{20} = 0$; $P_{21} = \frac{2}{3}\cdot\frac{2}{3}$; $P_{22} = \frac{1}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3}$; $P_{23} = \frac{1}{3}\cdot\frac{1}{3}$



$P_{30} = 0$; $P_{31} = 0$; $P_{32} = 1$; $P_{33} = 0$



$\therefore$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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Let X_n be rain state on day n.

then

$$P\{X_{n+1}=j \mid X_n, X_{n-1}, X_{n-2}\}$$

define $\tilde{X}_n \equiv \begin{pmatrix} X_n \\ X_{n-1} \\ X_{n-2} \\ \cancel{X_{n-3}} \end{pmatrix}$

Then $P\{\tilde{X}_{n+1} \mid \tilde{X}_n, \tilde{X}_{n-1}, \dots, \tilde{X}_2\} = P\{\hat{X}_{n+1} \mid \hat{X}_n\}$ or is a markov chain.

3 states are needed.

3 let $1 = \text{rain}$
$0 = \text{no rain}$

$$P\{X_{n+1}=1 \mid X_n = X_{n-1} = X_{n-2} = 1\} = .8$$

$$P\{X_{n+1}=0 \mid \quad \text{..} \quad \text{..} \quad \text{..} \quad \} = .2$$

$$P\{X_{n+1}= \quad \mid \sum X_a > 0\} =$$

States $\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}$

$\begin{array}{cccccccc} \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$

~~P(X_{n+2}~~

.6

$$X_n \left\{ \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & .6 & 0 & 0 & .4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & .6 & 0 & 0 & .4 & 0 & 0 \\ 2 & .6 & 0 & 0 & .4 & 0 & 0 & 0 & 0 \\ 3 & 0 & .4 & 0 & 0 & 0 & 0 & .6 & 0 \\ 4 & 0 & 0 & .6 & 0 & 0 & .4 & 0 & 0 \\ 5 & 0 & .4 & 0 & 0 & 0 & 0 & .6 & 0 \\ 6 & 0 & 0 & 0 & 0 & .4 & 0 & 0 & .6 \\ 7 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & .8 \end{array} \right.$$

$\underbrace{\hspace{6cm}}_{X_{n+1}}$

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(4) $$P\{X_{n+1}=j \mid X_n = i, X_{n-1}=i_{n-1}, \dots, X_0 = i_0\} = \begin{cases} P_{ij}^{I} & n \text{ even} \\ P_{ij}^{II} & n \text{ odd} \end{cases}$$

$i = 0,1,2$

Is $\{X_n, n \geq 0\}$ a markov chain? No, since Prob on Right hand side is Not fixed, but changes based on the time index.

$\Rightarrow$ $$P\{X_{2k+1}=j \mid X_{2k}=i, X_{2k-1}=i_{2k-1}, \dots, X_0=i_0\} = P_{ij}^{I}$$

& $$P\{X_{2k+2}=j \mid X_{2k+1}=i, \dots\} = P_{ij}^{II}$$

let $$\tilde{X}_n \equiv \begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix}$$

Then w/ state space $S = \{0,1,2,\bar{0},\bar{1},\bar{2}\}$ w/ state $i\,(\bar{i})$ signifies present value is i & present day is even/odd.

X_n:

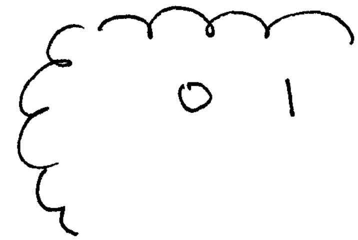
		0	1	2	$\bar{0}$	$\bar{1}$	$\bar{2}$
even	0	0	0	0	P_{00}^{I}	P_{01}^{I}	P_{02}^{I}
	1	0	0	0	P_{10}^{I}	P_{11}^{I}	P_{12}^{I}
	2	0	0	0	P_{20}^{I}	P_{21}^{I}	P_{22}^{I}
odd	$\bar{0}$	P_{00}^{II}	P_{01}^{II}	P_{02}^{II}	0	0	0
	$\bar{1}$	P_{10}^{II}	P_{11}^{II}	P_{12}^{II}	0	0	0
	$\bar{2}$	P_{20}^{II}	P_{21}^{II}	P_{22}^{II}	0	0	0

$\underbrace{0 \quad 1 \quad 2}_{\text{even}} \qquad \underbrace{\bar{0} \quad \bar{1} \quad \bar{2}}_{\text{odd}}$

X_{n+1}

(5)

$$\begin{array}{ccc} X & & Y \\ 0 & \longrightarrow & 0 \\ 1 & \longrightarrow & 1 \\ 2 & \nearrow & \end{array}$$



$\{Y_n, n \geq 0\}$ Markov chain? Yes

$$P\{Y_{n+1}=0 \mid Y_n=0\} = P\{X_{n+1}=0 \mid X_n=0\} = 0$$

$$P\{Y_{n+1}=1 \mid Y_n=0\} = P\{X_{n+1}=1 \text{ or } X_{n+1}=2 \mid X_n=0\} = 1$$

$$P\{Y_{n+1}=1 \mid Y_n=1\} = \frac{P\{Y_{n+1}=1, Y_n=1\}}{P\{Y_n=1\}}$$

$$= \frac{P\{X_{n+1}=1 \vee X_{n+1}=2 \wedge X_n=1 \vee X_n=2\}}{P\{X_n=1 \vee X_n=2\}}$$

$\pm$

$$P\{Y_{n+1}=0 \mid Y_n=1\} = \frac{1}{2}P_{10} + \frac{1}{2}P_{20} = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2} = \frac{3}{4}$$

$$\therefore P\{Y_{n+1}=1 \mid Y_n=1\} = \frac{1}{4}$$

$$P\{Y_{n+1}=1 \mid Y_n=1\} = \frac{1}{2}(P_{11}+P_{12}) + \frac{1}{2}(P_{21}+P_{22})$$

$$= \frac{1}{2}\left(\frac{1}{2}+0\right) + \frac{1}{2}(0+0) = \frac{1}{4} \quad \text{sense !!}$$

Thus Markov transition Matrix for $\{Y_n, n \geq 0\}$ is

$$Y_n \left\{ \begin{array}{c|c|c} & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 3/4 & 1/4 \\ \hline \end{array} \right.$$

$$\underbrace{\qquad\qquad}_{Y_{n+1}}$$

(6) $$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$P^2 = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} = \begin{bmatrix} p^2+(1-p)^2 & 2p(1-p) \\ 2p(1-p) & p^2+(1-p)^2 \end{bmatrix}$$

~~Check~~ PV: $$P^{(n)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^n & \frac{1}{2}-\frac{1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{1}{2}(2p-1)^n & \frac{1}{2}+\frac{1}{2}(2p-1)^n \end{bmatrix}$$

Check

$$P^{(1)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1) & \frac{1}{2}-\frac{1}{2}(2p-1) \\ \frac{1}{2}-\frac{1}{2}(2p-1) & \frac{1}{2}+\frac{1}{2}(2p-1) \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \checkmark$$

Assume true up to k.

$$P^{(k+1)} = P^{(1)}P^{(k)} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}\begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^k & \frac{1}{2}-\frac{1}{2}(2p-1)^k \\ \frac{1}{2}-\frac{1}{2}(2p-1)^k & \frac{1}{2}+\frac{1}{2}(2p-1)^k \end{bmatrix}$$

$$= \begin{bmatrix} \frac{p}{2}+\frac{p}{2}(2p-1)^k+\frac{(1-p)}{2}-\frac{(1-p)(2p-1)^k}{2} & \frac{p}{2}-\frac{p}{2}(2p-1)^k+\frac{(1-p)}{2}+\frac{(1-p)(2p-1)^k}{2} \\ \frac{(1-p)}{2}+\frac{(1-p)}{2}(2p-1)^k+\frac{p}{2}-\frac{p}{2}(2p-1)^k & \frac{(1-p)}{2}-\frac{(1-p)}{2}(2p-1)^k+\frac{p}{2}+\frac{p}{2}(2p-1)^k \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\left(p - (1-p)\right)(2p-1)^k & \frac{1}{2} + \left(-\frac{p}{2} + \frac{(1-p)}{2}\right)(2p-1)^k \\ \frac{1}{2} + \frac{1}{2}(1-p-p)(2p-1)^k & \frac{1}{2} + \frac{1}{2}\left(-(1-p)+p\right)(2p-1)^k \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1} & \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1} \\ \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1} & \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1} \end{bmatrix} \quad \checkmark$$

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(

7 $X_{n-1} = \begin{pmatrix} \text{rain yesterday} & = 0 \\ \text{rain day before yesterday} & = 0 \end{pmatrix}$

$X_n = \begin{pmatrix} \text{rain today} \\ \text{rain yesterday} \end{pmatrix}$

$X_{n+1} = \begin{pmatrix} \text{rain tomorrow} \\ \text{rain today} \end{pmatrix} = ?$

$$P = \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix}$$

$P\{\text{rain tomorrow}\} = P\{X = 0 \vee X = 1\}$

$$P^{(2)} = PP = \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix} \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix}$$

$$= \begin{bmatrix} .49 & .12 & .21 & .18 \\ .35 & .2 & .15 & .3 \\ .2 & .12 & .2 & .48 \\ .1 & .16 & .1 & .64 \end{bmatrix}$$

49
12, 18 $\rangle$ 30
21

$$P^2_{ij} = P\{X_{m+2} = j \mid X_m = i\}$$

If $X_m = 3$ $\quad P\{X_{m+2} = 0 \text{ or } X_{m+2} = 1 \mid X_m = 3\}$

$$= P\{X_{m+2} = 0 \mid X_m = 3\} + P\{X_{m+2} = 1 \mid X_m = 3\}$$

$$= .1 + .16 = .26$$

(8) let $X_n = 0$ if coin 0 is flipped on day n. $p_H = .7$

$X_n = 1$ " " 1 " " " " n. $p_H = .6$

$$P\{X_3 = 0 \mid X_0 = 1\} = ?$$

$$P\{X_{n+1} = 0 \mid X_n = 0\} = .7 \qquad P\{X_{n+1} = 0 \mid X_n = 1\} = .6$$

$$P\{X_{n+1} = 1 \mid X_n = 0\} = .3 \qquad P\{X_{n+1} = 1 \mid X_n = 1\} = .4$$

$$\therefore \quad P = \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}\begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix} = \begin{bmatrix} .49 + .18 & .21 + .12 \\ .42 + .24 & .18 + .16 \end{bmatrix} = \begin{bmatrix} .67 & .33 \\ .66 & .34 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}\begin{bmatrix} .67 & .33 \\ .66 & .34 \end{bmatrix} = \quad \ldots$$

Then read 1st row (1st elt is $P\{X_3 = 0 \mid X_0 = 0\}$)

(2nd elt is $\quad = 1 \mid \quad = \}$)

(9) P^r has all positive entries

$$P^n = P^r \cdot P^{(n-r)} =$$

$$(P^n)_{ij} = \sum_{k=0}^{\infty} P^r_{ik} P^{n-r}_{kj}$$

well

$$(P^{r+1})_{ij} = \sum_{k=0}^{\infty} P^r_{ik} P_{kj} \geq 0$$ as long as $P_{kj} \neq 0 \ \forall \ k$

But this must be true...

$\therefore$ By induction

(15) State i is recurrent if $f_i = 1$
transient if $f_i < 1$

f_i = prob starting in state i we will ever return to state i

P_1 States $\{0,1,2\}$ ~~ar~~ are a class & this Markov chain is irreducible, all states are recurrent.

P_2 classes ~~{0,1,2}{2}{2}~~ $\{0,1,2,3\}$
M.C. is irreducible all states are recurrent

P_3 $\{3,4\}$, $\{1,2,3\}$ All states are recurrent

P_4 $\{0,1\}$, $\{2\}$, $\{3\}$, $\{4\}$
0 & 1 are recurrent, 2 recurrent, 3 transient, 4 transient

(11) State j can be reach... $P_{ij}^{n} > 0$ for some n.

Pr: $P_{ij}^{n^*} > 0$ for $n^* < M$.

If $n < M$ in the choice (hypothesis) we are done.

Assume $n^* > M$ + Assume $P_{ij}^{k} = 0 \quad \forall \quad k \in [1, 2, \cdots, M]$

Then $P^{n} = P^{M l_1 + l_0}$

$$l_1 \in [1, 2 \cdots \infty] \quad l_0 \in [1, 2, \cdots, M-1]$$

Now

$$(P^{n})_{ij} = (P^{M l_1})(P^{l_0})$$

How do?

(12) i is recurrent $F_i = 1$ ~~$\Rightarrow$~~ $\sum_{n=1}^{\infty} p_{ii}^n = \infty$

$i \not\leftrightarrow j \Rightarrow P_{ij} = 0$

Assume that $P_{ij} \neq 0$ Then

~~If $P_{ij} > 0$ $P_{ji}^n = 0 \ \forall n$ o else~~

Since $P_{ij}^1 > 0$ + if $P_{ji}^n > 0$ then $i + j$ would communicate in contradiction.

So $P_{ji}^n = 0 \ \forall n$

But then if $P_{ij} > 0$ the process has a positive prob. of leaving i once entered (prob $= P_{ij} > 0$) This contradicts the recurrence of state i. $\therefore P_{ij} \not> 0$ + + never returning

$P_{ij} = 0$

(13) Random walk is transient when $p \neq 1/2$

$$P_{i,i+1} = p = \cancel{\quad} 1 - P_{i,i-1}$$

Following hint state $= \sum_{i=1}^{n} X_i$ this is the state after n timesteps

$P\{X_i = 1\} = p =$ X_i's are Bernoulli random variables.

If $p > 1/2$ then

$$E\left[\sum X_i\right] = \sum E\{X_i\} = \sum p(1) + (1-p)(-1) = \sum 2p - 1 \xrightarrow[n\to\infty]{} \infty$$

$p \neq +1/2$.

Thus since the avg elt goes to ∞. it can return to 0

only finitly many times $\Rightarrow$ State 0 is transient.

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(14) ~~distribution~~ prob to change states is geometric $\Rightarrow$ $P_n = (1-p)^{n-1} p$

State 1 $\Rightarrow$ Flipping coin 1

State 2 = Flipping coin 2

$P_{11} = .6 \qquad P_{12} = .4$

$P_{21} = .5 \qquad P_{22} = .5$

$$P = \begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix}\begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix} = \begin{bmatrix} .36 + .2 & .24 + .2 \\ .3 + .25 & .2 + .25 \end{bmatrix}$$

$$= \begin{bmatrix} .56 & .44 \\ .55 & .45 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} .56 & .44 \\ .55 & .45 \end{bmatrix}\begin{bmatrix} .56 & .44 \\ .55 & .45 \end{bmatrix} = \begin{bmatrix} .3136 + .242 & .2464 + .198 \\ .308 + .2475 & .242 + .2025 \end{bmatrix}$$

$$= \begin{bmatrix} .5556 & .4444 \\ .5555 & .4445 \end{bmatrix}$$

looks like $P^{(\infty)} = \begin{bmatrix} .\bar{5} & .\bar{4} \\ .\bar{5} & .\bar{4} \end{bmatrix}$

$\therefore \quad \pi_1 = .\bar{5} \quad + \quad \pi_2 = .\bar{4}$

so ~~pe~~ proportion of flips that use coin 1 is $\pi_1 = .\bar{5}$

(b) Start w/ coin = 1 desire to know $P_{12}^{5} = ?$