$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}-\bar{u}\Delta t}^{x_{i-1/2}} \left(Q_{i-1}^n + \sigma_{i-1}^n(\xi - x_{i-1})\right) d\xi$$

$$+ \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}-\bar{u}\Delta t} \left(Q_i^n + \sigma_i^n(\xi - x_i)\right) d\xi$$

$$= \frac{1}{\Delta x}\left[Q_{i-1}^n(\bar{u}\Delta t) + \sigma_{i-1}^n \frac{(\xi - x_{i-1})^2}{2}\Bigg|_{x_{i-1/2}-\bar{u}\Delta t}^{x_{i-1/2}} \right]$$

$$+ \frac{1}{\Delta x}\left[Q_i^n(\Delta x - \bar{u}\Delta t) + \sigma_i^n \frac{(\xi - x_i)^2}{2}\Bigg|_{x_{i-1/2}}^{x_{i+1/2}-\bar{u}\Delta t} \right]$$

$$= \frac{1}{\Delta x}\left[Q_{i-1}^n \bar{u}\Delta t + \frac{\sigma_{i-1}^n}{2}\left[(x_{i-1/2} - x_{i-1})^2 - (x_{i-1/2} - \bar{u}\Delta t - x_{i-1})^2\right]\right]$$

$$+ \frac{1}{\Delta x}\left[Q_i^n(\Delta x - \bar{u}\Delta t) + \frac{\sigma_i^n}{2}\left[(x_{i+1/2} - \bar{u}\Delta t - x_i)^2 - (x_{i-1/2} - x_i)^2\right]\right]$$

$$= \frac{1}{\Delta x}\left[Q_{i-1}^n \bar{u}\Delta t + \frac{\sigma_{i-1}^n}{2}\left[\left(\frac{\Delta x}{2}\right)^2 - \left(\frac{\Delta x}{2} - \bar{u}\Delta t\right)^2\right]\right]$$

$$+ \frac{1}{\Delta x}\left[Q_i^n(\Delta x - \bar{u}\Delta t) + \frac{\sigma_i^n}{2}\left[\left(\frac{\Delta x}{2} - \bar{u}\Delta t\right)^2 - \frac{\Delta x^2}{4}\right]\right]$$

$$= \cancel{\frac{\Delta}{\Delta}}\, Q^n_{i-1} \frac{\bar{U}\Delta t}{\Delta x} + \frac{\sigma^n_{i-1}}{2\Delta x}\left[\frac{\Delta x^2}{4} - \left(\frac{\Delta x^2}{4} - \Delta x \bar{U}\Delta t + \bar{U}^2 \Delta t^2 \right) \right]$$

$$+ \frac{Q^n_i}{\Delta x}(\Delta x - \bar{U}\Delta t) + \frac{\sigma^n_i}{2\Delta x}\left[\cancel{\frac{\Delta x^2}{4}} - \Delta x \bar{U}\Delta t + \bar{U}^2 \Delta t^2 - \cancel{\frac{\Delta x^2}{4}} \right]$$

$$= \cancel{Q^n_{i-1} \Delta t}\; Q^n_{i-1} \frac{\bar{U}\Delta t}{\Delta x} + \frac{\sigma^n_{i-1}}{2\Delta x}\left[\Delta x \bar{U}\Delta t - \bar{U}^2 \Delta t^2 \right]$$

$$+ Q^n_i\left(1 - \frac{\bar{U}\Delta t}{\Delta x}\right) + \frac{\sigma^n_i}{2\Delta x}\left[\underbrace{-\Delta x \bar{U}\Delta t + \bar{U}^2 \Delta t^2}_{} \right]$$

$$= \cancel{Q^n_{i-1} \frac{\bar{U}\Delta t}{\Delta x}} \qquad - \bar{U}\Delta t\left(+\Delta x - \bar{U}\Delta t\right)$$

$$= -\Delta x \bar{U}\Delta t\left(1 - \frac{\bar{U}\Delta t}{\Delta x}\right)$$

$$= \cancel{\frac{\bar{U}\Delta t}{\Delta x}\left[Q^n_{i-1} \right.}$$

$$= \frac{\bar{U}\Delta t}{\Delta x} Q^n_{i-1} + \bar{U}\Delta t \frac{\sigma^n_{i-1}}{2} - \frac{\bar{U}^2 \Delta t^2}{2\Delta x} \sigma^n_{i-1}$$

$$+ \left(1 - \frac{\bar{U}\Delta t}{\Delta x}\right) Q^n_i - \bar{U}\Delta t \frac{\sigma^n_i}{2} + \frac{\bar{U}^2 \Delta t^2}{2\Delta x} \sigma^n_i$$

$$= \frac{\bar{U}\Delta t}{\Delta x}\left(Q^n_{i-1} + \frac{1}{2}(\Delta x - \bar{U}\Delta t)\, \sigma^n_{i-1} \right)$$

$$+ \left(1 - \frac{\bar{U}\Delta t}{\Delta x}\right)\left[Q^n_i - \frac{\sigma^n_i}{2} \bar{U}\Delta t \right]$$

✓

(6.7) $\bar{u} < 0$

Then 6.1 to flux-limiter method

6.41 $$Q_i^{n+1} = Q_i^n - \nu(Q_{i+1}^n - Q_i^n)$$
$$+ \frac{1}{2}\nu(1+\nu)\left[\phi(\theta_{i+1/2}^n)(Q_{i+1}^n - Q_i^n) - \phi(\theta_{i-1/2}^n)(Q_i^n - Q_{i-1}^n)\right] \checkmark$$

Thm 6.1

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n(Q_i^n - Q_{i-1}^n) + D_i^n(Q_{i+1}^n - Q_i^n)$$

write 6.41 in the form required for Thm 6.1, 6.41 becomes

$$Q_i^{n+1} = Q_i^n + \left[-\nu + \frac{1}{2}\nu(1+\nu)\left[\phi(\theta_{i+1/2}^n) - \phi(\theta_{i-1/2}^n)\left(\frac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n}\right)\right]\right](Q_{i+1}^n - Q_i^n)$$

$\rightarrow C_{i-1}^n \equiv 0$ &

$$D_i^n = -\nu + \frac{1}{2}\nu(1+\nu)\left(\phi(\theta_{i+1/2}^n) - \phi(\theta_{i-1/2}^n)\left(\frac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n}\right)\right)$$

Now if $\bar{u} < 0$ Then $\theta_{i-1/2}^n \equiv \dfrac{\Delta Q_{I-1/2}^n}{\Delta Q_{i-1/2}^n} = \dfrac{\Delta Q_{i+1-1/2}^n}{\Delta Q_{i-1/2}^n} = \dfrac{\Delta Q_{i+1/2}^n}{\Delta Q_{i-1/2}^n}$

Thus $$= \frac{Q_{i+1}^n - Q_i^n}{Q_i^n - Q_{i-1}^n}$$

$$D_i^n = -\nu + \frac{1}{2}\nu(1+\nu)\left[\phi(\theta_{i+1/2}^n) - \frac{\phi(\theta_{i-1/2}^n)}{\theta_{i-1/2}^n}\right]$$

$C^n_{i+1} \geq 0 \quad \forall i$ ✓

$D^n_i \geq 0 \quad \forall i$

$\& \quad C^n_i + D^n_i \leq 1 \quad \forall i \Rightarrow \quad D^n_i \leq 1$

$\therefore \quad 0 \leq D^n_i \leq 1 \quad \forall i$

$$\Rightarrow \quad 0 \leq -\nu + \tfrac{1}{2}\nu(1+\nu)\left[\phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}}\right] \leq 1$$

$$\Rightarrow \frac{2\nu}{\nu(1+\nu)} \leq \phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}} \leq \frac{2(1+\nu)}{\nu(1+\nu)}$$

$$\Rightarrow \quad \frac{2}{(1+\nu)} \leq \phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}} \leq \frac{2}{\nu}$$

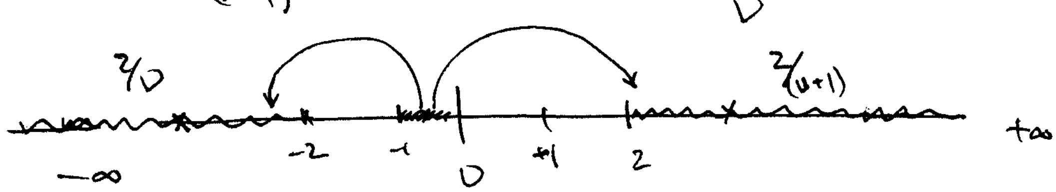
If the CFL condition holds $-1 \leq \nu \leq 0$

Then

$0 \leq \nu + 1 \leq 1$ $\qquad -1 \geq \frac{1}{\nu} \geq -\infty$

$1 \leq \frac{1}{\nu+1} \leq +\infty$ $\qquad -\infty \leq \frac{1}{\nu} \leq -1$

$2 \leq \frac{2}{(\nu+1)} \leq +\infty$ $\qquad -\infty \leq \frac{2}{\nu} \leq -2$



If we require that

$$2 \leq \left| \phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i+1/2})}{\theta^n_{i-1/2}} \right|$$

The above will always be satisfied

Require $2 \leq \left| \phi(\theta_1) - \frac{\phi(\theta_2)}{\theta_2} \right| \quad \forall\ \theta_1, \theta_2$

If we require that $2 \leq \phi(\theta) \leq \infty$

$$\leq \frac{\phi(\theta)}{\theta}$$

? Don't seem to be getting the correct results?

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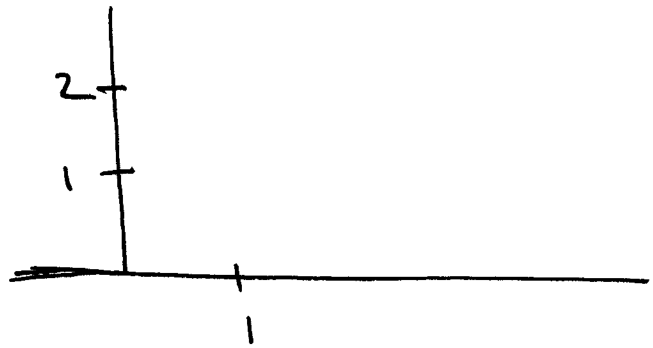
6.10 6.39 b Sweby region

van-Leer: $\phi(\theta) = \dfrac{\theta + |\theta|}{1 + \theta}$



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(8.1) $q_t + \bar{u} q_x = 0$

eq 4.19 is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}\left[\mathcal{F}(Q_{i+1}^n) - \mathcal{F}(Q_{i-1}^n)\right]$$

w/ Advection eq $\mathcal{F}(q) = \bar{u} q$ so the centered method become

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}\bar{u}\left[Q_{i+1}^n - Q_{i-1}^n\right] \quad *$$

with von-Neumann stability analysis

$$Q_I^n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{Q}^n(\xi) e^{i\xi I\Delta x}\, d\xi$$ putting this into the RHS of * gives

$$\hat{Q}_I^{n+1}(\xi) = \hat{Q}_I^n(\xi) - \tfrac{1}{2}\nu\left[e^{i\xi\Delta x} - e^{-i\xi\Delta x}\right]\hat{Q}_I^n(\xi) \qquad \text{w/ } \nu \equiv \frac{\bar{u}\Delta t}{\Delta x}$$

$$= \left(1 - \frac{\nu}{2}\cdot 2i \sin(\xi\Delta x)\right)\hat{Q}_I^n(\xi)$$

$$= \underbrace{\left(1 - i\nu\sin(\xi\Delta x)\right)}_{g(\xi,\Delta x,\Delta t)}\hat{Q}_I^n(\xi)$$

The amplitude factor in this case is of norm

$$|g(\xi,\Delta x,\Delta t)| = 1 + \nu^2 \sin^2(\xi\Delta x) > 1 \quad \forall \quad \nu > 0 \quad \therefore \text{ the method is}$$

unstable

(8.2) Upwind method eq 4.30

$$Q_i^{n+1} = Q_i^n - \underbrace{\frac{\bar{u}\,\Delta t}{\Delta x}}_{=\,\nu}\left(Q_{i+1}^n - Q_i^n\right)$$

$$\Rightarrow \quad Q_i^{n+1} = Q_i^n - \nu Q_{i+1}^n + \nu Q_i^n = (1+\nu)Q_i^n - \nu Q_{i+1}^n$$

Using the "1" norm $\|\cdot\|_1 = \Delta x \sum_i |Q_i|$

we get

$$\|Q^{n+1}\|_1 = \Delta x \sum_i \left|(1+\nu)Q_i^n - \nu Q_{i+1}^n\right|$$

$$\le |1+\nu|\, \underbrace{\Delta x \sum_i Q_i^n}_{\|Q^n\|_1} + |\nu|\, \underbrace{\Delta x \sum_i Q_{i+1}^n}_{\|Q^n\|_1}$$

if $1+\nu \geq 0 \Rightarrow \nu \geq -1$

& $\nu < 0$

$|\nu| = -\nu$

$$= (1+\nu+|\nu|)\|Q^n\|_1$$

$$= 1 \cdot \|Q^n\|_1$$

$\therefore$ We must require $-1 \le \nu < 0$ which is condition 4.32

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(8.3) $q_t + \bar{u} q_x = a q \qquad q(x,0) = \overset{\circ}{q}(x)$

w/ solution $q(x,t) = e^{at}\, \overset{\circ}{q}(x - \bar{u}t)$

(a) Show

$$Q_i^{n+1} = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) + \Delta t\, a\, Q_i^n$$

$$\tau_i \equiv \frac{1}{\Delta t}\left[\mathcal{N}(q^n) - q^{n+1}\right]$$

$$= \frac{1}{\Delta t}\left[q(x_i,t_n) - \frac{\bar{u}\Delta t}{\Delta x}\left(q(x_i,t_n) - q(x_{i-1},t_n)\right) + \Delta t\, a\, q(x_i,t_n) - q(x_i,t_{n+1})\right]$$

$$= \frac{1}{\Delta t}\Big[\cancel{q(x_i,t_n)} - \frac{\bar{u}\Delta t}{\Delta x}\Big(\cancel{q(x_i,t_n)} - \big(\cancel{q(x_i,t_n)} - \Delta x\, q_x(x_i,t_n) + \frac{\Delta x^2}{2} q_{xx}(\cdot,\cdot) + O(\Delta x^3)\big)\Big)$$

$$+ \Delta t\, a\, q(\cdot,\cdot) - \Big(\cancel{q(x_i,t_n)} + \Delta t\, q_t(x_i,t_n) + \frac{\Delta t^2}{2} q_{tt}(\cdot,\cdot) + O(\Delta t^3)\Big)\Big]$$

$$= \frac{1}{\cancel{\Delta t}}\left[+\frac{\bar{u}\cancel{\Delta t}}{\Delta x}\right]\left[-\Delta x\, q_x(\cdot,\cdot) + \frac{\Delta x^2}{2} q_{xx}(\cdot,\cdot) + O(\Delta x^3)\right]$$

$$+ a q(\cdot,\cdot) - q_t(\cdot,\cdot) + \frac{\Delta t}{2} q_{tt}(\cdot,\cdot) + O(\Delta t^2)$$

$$= -\bar{u} q_x(\cdot,\cdot) + \frac{\bar{u}}{2}\Delta x\, q_{xx}(\cdot,\cdot) + O(\Delta x^2) - q_t + a q + \frac{\Delta t}{2} q_{tt} + O(\Delta t^2)$$

$$= -\underbrace{\left(q_t + \bar{u} q_x - a q\right)}_{\equiv 0} + \frac{\bar{u}\Delta x}{2} q_{xx} + \frac{\Delta t}{2} q_{tt} + O(\Delta x^2) + O(\Delta t^2)$$

Since ~~[illegible]~~ $\nu = \frac{\bar{u}\Delta t}{\Delta x}$ & $\nu = O(1)$

$\Delta t = O(\Delta x)$ & we have a 1st order method

(b) Show $\|\mathcal{N}(E^n)\| \le (1+\alpha\Delta t)\|E^n\|$

For any grid function E^n.

$$\mathcal{N}(E^n) = E_i^n - \nu(E_i^n - E_{i-1}^n) + \Delta t a E_i^n$$

$$= (1-\nu)E_i^n + \nu E_{i-1}^n + \Delta t a E_i^n$$

$$\|\mathcal{N}(E^n)\|_1 \le |1-\nu|\,\|E^n\|_1 + |\nu|\,\|E^n_1\| + \Delta t|a|\,\|E^n\|_1$$

$$= (|1-\nu| + |\nu| + \Delta t|a|)\|E^n\|_1$$

If $1-\nu > 0$ & $\nu > 0$

$|1-\nu| = 1-\nu$ & $|\nu| = \nu$ so the above is true $0<\nu<1$

$$\|\mathcal{N}(E^n)\|_1 \le (1+\Delta t a)\|E^n\|_1 \quad \checkmark$$

(c) TVB if $\exists$ constant R & $TV(Q^n) < R$ $\forall$ $n\Delta t \le T$ & $\Delta t < \Delta t_0$.

$$TV(Q^n) = \sum_i |Q_i^n - Q_{i-1}^n|$$

$$TV(Q^{n+1}) = \sum_i |Q_i^{n+1} - Q_{i-1}^{n+1}| = \cancel{\sum_i \ldots}$$

$$= \sum_i \left| Q_i^n - \nu(Q_i^n - Q_{i-1}^n) + \Delta t a Q_i^n - \right.$$

$$-\left(Q^n_{i-1} - \nu(Q^n_{i-1} - Q^n_{i-2}) + \Delta t a Q^n_{i-1}\right)\Big|$$

$$= \sum_i \left| Q^n_i - Q^n_{i-1} - \nu(Q^n_i - Q^n_{i-1} - (Q^n_{i-1} - Q^n_{i-2})) + \Delta t a(Q^n_i - Q^n_{i-1})\right|$$

$$= \sum_i \left| Q^n_i - Q^n_{i-1} - \nu(Q^n_i - Q^n_{i-1}) + \nu(Q^n_{i-1} - Q^n_{i-2}) + \Delta t a(Q^n_i - Q^n_{i-1})\right|$$

$$= \sum_i \left| (1-\nu)(Q^n_i - Q^n_{i-1}) + \nu(Q^n_{i-1} - Q^n_{i-2}) + \Delta t a(Q^n_i - Q^n_{i-1})\right|$$

$$\le \sum_i \left((1-\nu)|Q^n_i - Q^n_{i-1}| + \nu|Q^n_{i-1} - Q^n_{i-2}| + \Delta t|a|\,|Q^n_i - Q^n_{i-1}|\right)$$

Since $0 < \nu < 1$

$$\le (1-\nu)TV(Q^n) + \nu TV(Q^n) + \Delta t|a|\, TV(Q^n)$$

$$= \cancel{TV(Q^n)}\ (1 + \Delta t|a|)\, TV(Q^n)$$

$$\therefore \quad TV(Q^{n+1}) \le (1+\Delta t|a|)\, TV(Q^n)$$

So $TV(Q^1) \le (1+\Delta t|a|)\, TV(Q^0)$

$TV(Q^2) \le (1+\Delta t|a|)^2\, TV(Q^0)$

$\vdots$

$$TV(Q^n) \le (1+\Delta t|a|)^n\, TV(Q^0) \le \left(e^{\Delta t|a|}\right)^n TV(Q^0)$$

$\therefore \quad TV(Q^n) \leq e^{T|a|} TV(Q^0)$

Then define $R = e^{T|a|} TV(Q^0)$ + this scheme is total-variation bounded.

To answer the question if it is Total variation diminishing TVD.

consider the theorem of Harten: Given

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n (Q_i^n - Q_{i-1}^n) + D_i^n (Q_{i+1}^n - Q_i^n)$$

Then $TV(Q^{n+1}) \leq TV(Q^n)$

provided $C_{i-1}^n \geq 0$

$D_i^n \geq 0$

+ $C_i^n + D_i^n \leq 1$

this won't work for this scheme since we have no D_i^n type term.

If $a = 0$ this method will work to show

Since $C_{i-1}^n \equiv \frac{\bar{u}\Delta t}{\Delta x} \geq 0$ ✓

$D_i^n \equiv 0 \geq 0$ ✓

+ $C_i^n + D_i^n = \frac{\bar{u}\Delta t}{\Delta x} \leq 1$ is the CFL condition so the method

is TVD.

~~If $a < 0$ we expect this to be true but I don't know how to~~

~~show it~~

~~If $a > 0$~~

Since we obtained

$TV(Q^{n+1}) \le (1+\Delta t|a|)\,TV(Q^n)$ I can't conclude that the method

satisfies

$TV(Q^{n+1}) \le TV(Q^n)$. $\therefore$ In general no, this method is Not TVD.

(B.4) $\|\mathcal{N}(P)-\mathcal{N}(Q)\|_1 \le \|P-Q\|_1$ Must be TVD

consider

~~[illegible]~~ $TV(\mathcal{N}(Q)) \le TV(Q)$

let $P_i^n \equiv Q_i^n$ + $Q \equiv Q_{i-1}^n$

Then $\|\mathcal{N}(P)-\mathcal{N}(Q)\|_1 = \Delta x \sum_i |Q_i^{n+1} - Q_{i-1}^{n+1}| = \Delta x\, TV(Q^{n+1})$

$$\le \|P-Q\|_1 = \Delta x \sum_i |Q_i^n - Q_{i-1}^n| = \Delta x\, TV(Q^n).$$

$\uparrow$

By hypothesis

$\therefore$ $TV(Q^{n+1}) \le TV(Q^n)$.

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(8.5) Harten's Theorem is given

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n(Q_i^n - Q_{i-1}^n) + D_i^n(Q_{i+1}^n - Q_i^n)$$

Then $TV(Q^{n+1}) \le TV(Q^n)$ provided

$$C_{i-1}^n \geq 0$$

$$D_i^n \geq 0$$

$$+ \quad C_i^n + D_i^n \le 1$$

Consider

$$\begin{aligned}
Q_i^{n+1} - Q_{i-1}^{n+1} &= Q_i^n - C_{i-1}^n(Q_i^n - Q_{i-1}^n) + D_i^n(Q_{i+1}^n - Q_i^n) \\
&\quad - Q_{i-1}^n + C_{i-2}^n(Q_{i-1}^n - Q_{i-2}^n) - D_{i-1}^n(Q_i^n - Q_{i-1}^n) \\
&= Q_i^n - Q_{i-1}^n - C_{i-1}^n(Q_i^n - Q_{i-1}^n) - D_{i-1}^n(Q_i^n - Q_{i-1}^n) \\
&\quad + D_i^n(Q_{i+1}^n - Q_i^n) + C_{i-2}^n(Q_{i-1}^n - Q_{i-2}^n) \\
&= (1 - C_{i-1}^n - D_{i-1}^n)(Q_i^n - Q_{i-1}^n) \\
&\quad + D_i^n(Q_{i+1}^n - Q_i^n) + C_{i-2}^n(Q_i^n - Q_{i-1}^n)
\end{aligned}$$

Thus

$$\sum_i |Q_i^{n+1} - Q_{i-1}^{n+1}| \le \sum_i (1 - C_{i-1}^n - D_{i-1}^n)|Q_i^n - Q_{i-1}^n| + D_i^n|Q_{i+1}^n - Q_i^n| + C_{i-2}^n|Q_i^n - Q_{i-1}^n|$$

By way of the signs of each term in the hypothesis

since $C_i^n + D_i^n \le 1$ & $C_i^n \geq 0$ & $D_i^n \geq 0$

$$0 \le 1 - C_i^n - D_i^n \le 1$$

since this implys $C_i^n + D_i^n \geq 0$ which is true

$\therefore$

$$TV(Q^{n+1}) \le \sum_i (1 - C_{i-1}^n - D_{i-1}^n)|Q_i^n - Q_{i-1}^n| + \sum_i D_i^n |Q_{i+1}^n - Q_i^n|$$

$$+ \sum_i C_{i-2}^n |Q_i^n - Q_{i-1}^n|$$

Now shift the indices of the 1st sum up by 1 & the

$$(1 - C_{i-1}^n - D_{i-1}^n)|Q_i^n - Q_{i-1}^n| \Rightarrow (1 - C_i^n - D_i^n)|Q_{i+1}^n - Q_i^n|$$

sim. shift the indices of the last ~~sum~~ sum by 2 up.

$$\rightarrow TV(Q^{n+1}) \le \sum_i (1 - \cancel{C_i^n} - \cancel{D_i^n})|Q_{i+1}^n - Q_i^n| + \cancel{D_i^n}|Q_{i+1}^n - Q_i^n| + \cancel{C_i^n}|Q_{i+1}^n - Q_i^n|$$

$$= TV(Q^n) \checkmark$$

(8.6) method 4.64 is

$$Q_i^{n+1} = Q_i^n - (Q_i^n - Q_{i-1}^n) - (\nu-1)(Q_{i-1}^n - Q_{i-2}^n)$$

$$= Q_{i-1}^n - (\nu-1)(Q_{i-1}^n - Q_{i-2}^n)$$

$$= (2-\nu)Q_{i-1}^n + (\nu-1)Q_{i-2}^n$$

If we can show $\mathcal{N}(Q) \equiv (2-\nu)Q_{i-1}^n - (\nu-1)Q_{i-2}^n$ is a contraction

~~[illegible]~~ mapping i.e. $\|\mathcal{N}(Q)-\mathcal{N}(P)\| \le (1+\alpha\Delta t)\|Q-P\|$ then the method is stable in this norm.

~~[illegible]~~ Since our $\mathcal{N}(\cdot)$ is also a linear operator it suffices to show

$$\|\mathcal{N}(E^n)\| \le (1+\alpha\Delta t)\|E^n\|$$

consider $\|\mathcal{N}(E^n)\|_1 = \|(2-\nu)E_{i-1}^n + (\nu-1)E_{i-2}^n\|_1 = \Delta x \sum_i |(2-\nu)E_{i-1}^n + (\nu-1)E_{i-2}^n|$

$$\le \Delta x \sum_i (2-\nu)|E_{i-1}^n| + (\nu-1)|E_{i-2}^n| \quad \text{iff} \quad 1 < \nu < 2$$

$$= \cancel{\Delta x \sum (2-\nu)\|E^n\|_1}$$

$$= (2-\nu)\|E^n\|_1 + (\nu-1)\|E^n\|_1 = \|E^n\|_1$$ ∴ the method is stable

(8.7) eq 8.41 is

$$Q_i^{n+1} = Q_i^n - \nu(Q_i^n - Q_{i-1}^n) \quad * \qquad \nu \equiv \frac{\bar{u}\Delta t}{\Delta x}$$

+ 8.44 is

$$v_t + \bar{u} v_x = \frac{1}{2}\bar{u}\Delta x(1-\nu)v_{xx}$$

The truncation error for eq * is

$$\tau^n = \frac{1}{\Delta t}\left[N(q^n) - q^{n+1}\right]$$

$$= \frac{1}{\Delta t}\left[q_i^n - \nu(q_i^n - q_{i-1}^n) - q_i^{n+1}\right]$$

$$= \frac{1}{\Delta t}\Big[\cancel{q_i^n} - \nu(\cancel{q_i^n} - (\cancel{q_i^n} - \Delta x \dot{q}_i^n + \frac{\Delta x^2}{2}\ddot{q}_i^n + O(\Delta x^3))) -$$

$$(\cancel{q_i^n} + \Delta t\, q_t(x_i,t_n) + \frac{\Delta t^2}{2} q_{tt}(x_i,t_n) + O(\Delta t^3))\Big]$$

$$= \frac{1}{\Delta t}\Big[+\nu(-\Delta x\, q_x(x_i,t_n) + \frac{\Delta x^2}{2} q_{xx}(x_i,t_n) - \frac{\Delta x^3}{6} q_{xxx}(x_i,t_n) + O(\Delta x^4))$$

$$\mp \Delta t\, q_t \mp \frac{\Delta t^2}{2} q_{tt} + O(\Delta t^3)\Big]$$

$$= \frac{1}{\Delta t}\Big(-\Delta t \bar{u}\, q_x + \frac{\Delta x \bar{u} \Delta t}{2} q_{xx} + O(\Delta x^3) \mp \Delta t\, q_t \mp \frac{\Delta t^2}{2} q_{tt} + O(\Delta t^3)\Big]$$

$$= \frac{1}{\Delta t}\Big(-\Delta t(q_t + \bar{u} q_x \mp \frac{\Delta x \bar{u}}{2} q_{xx} + \frac{\Delta t}{2} q_{tt}) + O(\Delta x^3) + O(\Delta t^3)\Big)$$

$$= \cancel{\frac{1}{\Delta t}} \; \cancel{q_t} + - \left[q_t + \bar{u} q_x \mp \frac{\Delta x \bar{u}}{2} q_{xx} + \frac{\Delta t}{2} q_{tt} + O(\Delta x^2) + O(\Delta t^2) \right]$$

Since q solves

$$q_t = \bar{u} q_x + \tfrac{1}{2} \bar{u} \Delta x (1-\nu) q_{xx}$$

$$q_{tt} = \bar{u} q_{tx} + \tfrac{1}{2} \bar{u} \Delta x (1-\nu) q_{xxt}$$

$$= \cancel{\frac{\partial}{\partial x}} \cancel{(\bar{u} q_t}$$

$$= \bar{u} \frac{\partial}{\partial x}(q_t) + \tfrac{1}{2} \bar{u} \Delta x (1-\nu) \frac{\partial^2}{\partial x^2}(q_t)$$

$$= \bar{u} \frac{\partial}{\partial x}\left(-\bar{u} q_x + \tfrac{1}{2} \bar{u} \Delta x (1-\nu) q_{xx}\right) + \tfrac{1}{2} \bar{u} \Delta x (1-\nu) \frac{\partial^2}{\partial x^2}\left(-\bar{u} q_x + \tfrac{1}{2} \bar{u} \Delta x (1-\nu) q_{xx}\right)$$

$$= -\bar{u}^2 q_{xx} + \tfrac{1}{2} \bar{u}^2 \Delta x (1-\nu) q_{xxx} \mp \tfrac{1}{2} \bar{u}^2 \Delta x (1-\nu) q_{xxx} + O(\Delta x^2)$$

$\therefore$

$$\tau^n = -\left[q_t + \bar{u} q_x \underbrace{- \frac{\Delta x \bar{u}}{2} q_{xx} + \frac{\Delta t}{2} \bar{u}^2 q_{xx}} + O(\Delta t^2) \right]$$

$$\underbrace{- \frac{\Delta x \bar{u}}{2}\Big(1 - \underbrace{\frac{\Delta t \bar{u}}{\Delta x}}_{\nu}\Big) q_{xx}}_{=0 \text{ By def } q} + O(\Delta t^2)$$

$\therefore \; \tau^n = O(\Delta t^2)$

(8.8) Lax-Wendroff method is:

$$Q_i^{n+1} = Q_i^n - \frac{\bar{u}\Delta t}{2\Delta x}\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 \bar{u}^2 \left(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n\right)$$

$$Q_i^{n+1} = Q_i^n - \frac{\nu}{2}\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{\nu^2}{2}\left(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n\right)$$

$\|$

$$\cancel{q} + q_t \Delta t + q_{tt}\frac{\Delta t^2}{2} + q_{ttt}\frac{\Delta t^3}{6} + O(\Delta t^4)$$

$$= q - \frac{\nu}{2}\Big(\cancel{q} + q_x \Delta x + \cancel{q_{xx}\frac{\Delta x^2}{2}} + q_{xxx}\frac{\Delta x^3}{6} + O(\Delta x^4)$$

$$- \Big(\cancel{q} - q_x \Delta x + \cancel{q_{xx}\frac{\Delta x^2}{2}} - q_{xxx}\frac{\Delta x^3}{6} + O(\Delta x^4)\Big)\Big)$$

$$+ \frac{\nu^2}{2}\Big(\cancel{q} - \cancel{q_x \Delta x} + q_{xx}\frac{\Delta x^2}{2} - \cancel{q_{xxx}\frac{\Delta x^3}{6}} + O(\Delta x^4)$$

$$+ 2q_{xxxx}\frac{\Delta x^4}{4!}$$

$$- \cancel{2q}$$

$$+ \cancel{q} + \cancel{q_x \Delta x} + q_{xx}\frac{\Delta x^2}{2} + \cancel{q_{xxx}\frac{\Delta x^3}{6}} + O(\Delta x^4)\Big)$$

$$\Rightarrow \quad q_t \Delta t + q_{tt}\frac{\Delta t^2}{2} + q_{ttt}\frac{\Delta t^3}{6} + O(\Delta t^4)$$

$$= -\frac{\nu}{2}\left[2q_x \Delta x + 2q_{xxx}\frac{\Delta x^3}{6} + O(\Delta x^5)\right]$$

$$+ \frac{\nu^2}{2}\left[q_{xx}\Delta x^2 + q_{xxxx}\frac{\Delta x^4}{12} + O(\Delta x^6)\right]$$

$$\Rightarrow \quad q_t \Delta t + q_{tt}\frac{\Delta t^2}{2} + q_{ttt}\frac{\Delta t^3}{6} + O(\Delta t^4)$$

$$= -\Delta t\, \bar{u}\, q_x - \frac{\Delta t\, \bar{u}\, \Delta x^2}{6} q_{xxx} + O(\Delta x^5)$$

$$+ \frac{\Delta t^2 \bar{u}^2}{2} q_{xx} + \frac{\Delta t^2 \bar{u}^2 \Delta x^2}{24} q_{xxxx} + O(\Delta x^6)$$

$$\Rightarrow \quad q_t + \bar{u} q_x = -\frac{\bar{u}\Delta x^2}{6} q_{xxx} + \frac{\Delta t\, \bar{u}^2}{2} q_{xx} + \frac{\Delta t\, \bar{u}^2 \Delta x^2}{24} q_{xxxx} + O(\Delta x^4)$$

$$- \frac{\Delta t}{2} q_{tt} - \frac{\Delta t^2}{6} q_{ttt} + O(\Delta t^3)$$

$$\therefore \quad q_t = -\bar{u} q_x + \frac{\Delta t\, \bar{u}^2}{2} q_{xx} - \frac{\Delta t}{2} q_{tt} - \frac{\bar{u}\Delta x^2}{6} q_{xxx} - \frac{\Delta t^2}{6} q_{ttt} + O(\Delta x^3)$$

Requires the evaluation of q_{tt} & q_{ttt} up to $O(\Delta x^1)$ & $O(1)$ respectively

$$\text{So} \quad q_{tt} = \underset{T_1}{-\bar{u} q_{xt}} + \underset{T_2}{\frac{\Delta t\, \bar{u}^2}{2} q_{txx}} - \underset{T_3}{\frac{\Delta t}{2} q_{ttt}} - \underset{T_4}{\frac{\bar{u}\Delta x^2}{6} q_{txxx}} - \frac{\Delta t^2}{6} q_{tttt} + O(\Delta x^3)$$

$$= -\bar{u}\left[-\bar{u} q_{xx} + \frac{\Delta t\, \bar{u}^2}{2} q_{xxx} - \frac{\Delta t}{2} q_{xtt} + O(\Delta^3)\right] + T_2, \dots, T_5$$

$$= \bar{u}^2 q_{xx} - \frac{\Delta t\, \bar{u}^3}{2} q_{xxx} + \frac{\bar{u}\Delta t}{2} q_{xtt} + O(\Delta^3)$$

$$\parallel$$

$$\frac{\bar{u}\Delta t}{2}\left[\bar{u}^2 q_{xxx} - \frac{\Delta t\, \bar{u}^3}{2} q_{xxxx} + O(\Delta)\right]$$

$$= \bar{u}^2 q_{xx} - \frac{\Delta t\, \bar{u}^3}{2} q_{xxx} + \frac{\bar{u}^3 \Delta t}{2} q_{xxx} - O(\Delta^2) \quad \checkmark$$

$$+ \frac{\Delta t\, \bar{u}^2}{2}\left[-\bar{u}\, q_{xxx} + O(\Delta)\right] \qquad \text{"}T_2\text{"}$$

$$+ -\frac{\Delta t}{2}\left[(-\bar{u})^3 q_{xxx} + O(\Delta)\right] \qquad \text{"}T_3\text{"}$$

$$- \cancel{\frac{\bar{u}\,\Delta x^2}{6}}\; O(\Delta x^2) + O(\Delta^2) \qquad T_4 + T_5$$

sim. $q_{ttt} = (-\bar{u})^3 q_{xxx} + O(\Delta)$

so

$$q_t = -\bar{u} q_x + \frac{\Delta t\, \bar{u}^2}{2}\cancel{q_{xx}} - \frac{\Delta t}{2}\left[\cancel{\bar{u}^2} q_{xx} - \cancel{\frac{\Delta t\, \bar{u}^3}{2}} q_{xxx} + \cancel{\bar{u}^3 \frac{\Delta t}{2}} q_{xxx} - \cancel{\bar{u}^3 \frac{\Delta t}{2}} q_{xxx} + \cancel{\frac{\Delta t\, \bar{u}^3}{2}} q_{xxx}\right]$$

$$- \frac{\bar{u}\,\Delta x^2}{6} q_{xxx} - \frac{\Delta t^2}{6}(-\bar{u})^3 q_{xxx} + O(\Delta^3)$$

$$q_t + \bar{u} q_x = \cancel{\Delta t\, \bar{u}^2} - \frac{\bar{u}\,\Delta x^2}{6} q_{xxx} + \frac{\Delta t^2 \bar{u}^3}{6} q_{xxx} + O(\Delta^3)$$

$$= -\frac{1}{6}\bar{u}\,\Delta x^2 (1 - \nu^2) q_{xxx} + O(\Delta^3) \qquad \text{eq } 8.45 \ \checkmark$$

(8.9) Eq 4.19 is

Can the stability be studied for general f?

$$Q^{n}_{i+1} = Q^{n}_{i} - \frac{\Delta t}{2\Delta x}\left(f(Q^{n}_{i+1}) - f(Q^{n}_{i-1})\right)$$

Consider the case when $f(q) = \bar{u} q$ & taylor expand

$$\cancel{q} + \cancel{\Delta t} q_t + \frac{\Delta t^2}{2} q_{tt} + O(\Delta^3) = \cancel{q} - \frac{\cancel{\Delta t}\,\bar{u}}{2\Delta x}\left[2\Delta x q_x + 2\frac{\Delta x^3}{3!} q_{xxx} + 2\frac{\Delta x^5}{5!} q_{xxxxx} + O(\Delta x^7)\right]$$

$$\Rightarrow \quad q_t + \frac{\Delta t}{2} q_{tt} + O(\Delta^2) = -\bar{u} q_x - \bar{u}\frac{\Delta x^2}{3!} q_{xxx} - \bar{u}\frac{\Delta x^4}{5!} q_{xxxxx} + O(\Delta x^6)$$

$$\Rightarrow \quad q_t + \bar{u} q_x = -\frac{\Delta t}{2} q_{tt} - \frac{\bar{u}\Delta x^2}{6} q_{xxx} - \bar{u}\frac{\Delta x^4}{5!} q_{xxxxx} + O(\Delta^6)$$

Now compute q_{tt} up to $O(\Delta)$

$$q_t = -\bar{u} q_x - \frac{\Delta t}{2} q_{tt} - O(\Delta^2)$$

$$q_{tt} = -\bar{u} q_{tx} - \frac{\Delta t}{2} q_{xtt} - O(\Delta^2)$$

Now q_{tx} is $\frac{\partial}{\partial x}$ of q_t

$$= \cancel{-\bar{u}\left[-\bar{u} q_{xx} - \frac{\Delta t}{2} q_{xtt}\right] - \frac{\Delta t}{2}}\left[+\bar{u}^2 q_{xxx} + O(\Delta)\right]$$

& q_{xtt} is $\frac{\partial}{\partial x}$ of q_{tt}

$$= \cancel{\bar{u}^2 q_{xx} + \frac{\Delta t}{2}\bar{u} q_{xtt} - \frac{\Delta t}{2}\bar{u}^2 q_{xxx} + O(\Delta^2)}$$

$$= -\bar{u}\left[-\bar{u} q_{xx} - \frac{\Delta t}{2} q_{xtt} + O(\Delta^2)\right] \cancel{- \frac{\Delta t}{2}\left[\bar{u} q_{xxx} + O(\Delta)\right]}$$

$$- \frac{\Delta t}{2}\left[\bar{u}^2 q_{xxx} + O(\Delta) \right]$$

$$= + \bar{u}^2 q_{xx} + \underbrace{\frac{\bar{u}\Delta t}{2} q_{xtt}}_{\frac{\bar{u}\Delta t}{2}\left[\bar{u}^2 q_{xxx}\right]} - \frac{\bar{u}^2 \Delta t}{2} q_{xxx} + O(\Delta^2)$$

$$\therefore \quad q_{tt} = \bar{u}^2 q_{xx} + \frac{\bar{u}^3 \Delta t}{2} q_{xxx} - \frac{\bar{u}^2 \Delta t}{2} q_{xxx} + O(\Delta^2)$$

Then the modified eq is

$$\cancel{q_t + \bar{u} q_x - \frac{\Delta t}{2}}$$

$$q_t + \bar{u} q_x = -\frac{\Delta t}{2}\left[\bar{u}^2 q_{xx} + \frac{\bar{u}^3 \Delta t}{2} q_{xxx} - \frac{\bar{u}^2 \Delta t}{2} q_{xxx} + O(\Delta^2) \right]$$

$$- \bar{u} \frac{\Delta x^2}{6} q_{xxx}$$

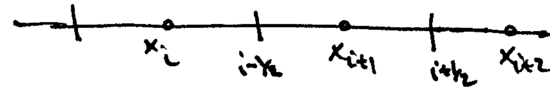
$$= -\frac{\Delta t}{2} \bar{u}^2 q_{xx} + O(\Delta t^2)$$

$\therefore \quad q_t + \bar{u} q_x = \beta q_{xx} \quad$ w/ $\quad \beta = -\frac{\Delta t}{2} \bar{u}^2 < 0 \quad \therefore$ this method is unstable

9.1 7.18 is Color Eq: $\boxed{\bar{q}_t + u(x)\bar{q}_x = 0}$

$$\bar{Q}_i^{n+1} = \bar{Q}_i^n - \frac{\Delta t}{\Delta x}\left(A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$$

w/ $\tilde{F}_{i-1/2} = \frac{1}{2}|s_{i-1/2}|\left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}|\right) W_{i-1/2}$



noteino limiter

here $W_{i-1/2} = \bar{Q}_i - \bar{Q}_{i-1}$ & $s_{i-1/2} = \begin{cases} u_i & \bar{u} > 0 \\ u_{i-1} & \bar{u} < 0 \end{cases}$

& $A^+\Delta\bar{Q}_{i-1/2} = s^+_{i-1/2} W_{i-1/2}$

$A^-\Delta\bar{Q}_{i-1/2} = s^-_{i-1/2} W_{i-1/2}$ So with these substitutions we

obtain: Assuming $\bar{u} > 0$

$$\bar{Q}_i^{n+1} = \bar{Q}_i^n - \frac{\Delta t}{\Delta x}\left(s^+_{i-1/2}(\bar{Q}_i^n - \bar{Q}_{i-1}^n) + s^-_{i+1/2}(\bar{Q}^n_{i+1} - \bar{Q}^n_i)\right)$$

$$- \frac{\Delta t}{\Delta x}\frac{1}{2}\Big[|u_{i+1}|\left(1 - \frac{\Delta t}{\Delta x}|u_{i+1}|\right)(\bar{Q}^n_{i+1} - \bar{Q}^n_{i})$$

$$- |u_i|\left(1 - \frac{\Delta t}{\Delta x}|u_i|\right)(Q^n_i - Q^n_{i-1})\Big]$$

✓

Again Assuming $u_i > 0$ & $u_{i+1} > 0$ one obtains

$$\tau^n = \frac{1}{\Delta t}\left[N(q^n) - q^{n+1}\right]$$

Method is

$$s^+_{i-1/2} = \max(s_{i-1/2}, 0)$$
$$s^-_{i-1/2} = \min(s_{i-1/2}, 0)$$

$$\bar{Q}_i^{n+1} = \bar{Q}_i^n - \frac{\Delta t}{\Delta x}\left[u_i(\bar{Q}_i^n - \bar{Q}_{i-1}^n)\right]$$
$$- \frac{\Delta t}{2\Delta x}\left[u_{i+1}\left(1 - \frac{\Delta t}{\Delta x}u_{i+1}\right)(\bar{Q}^n_{i+1} - \bar{Q}^n_i) - u_i\left(1 - \frac{\Delta t}{\Delta x}u_i\right)(Q^n_i - Q^n_{i-1})\right] \quad *$$

Taylor expanding this result gives: (The RHS) about x_i

$$q - \frac{\Delta t}{\Delta x}\left(u\left(\cancel{q} - \cancel{q} + \Delta t q_t - \frac{\Delta t^2}{2}q_{tt} + \frac{\Delta t^3}{6}q_{ttt} + O(\Delta t^4)\right)\right)$$
$$- \frac{\Delta t}{2\Delta x}\left[\left(u + \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} + \frac{\Delta x^3}{6}u_{xxx} + O(\Delta x^4)\right)\left(1 - \frac{\Delta t}{\Delta x}\left(u + \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} + O(\Delta x^3)\right)\right)\right.$$
$$\cdot\left(\cancel{q} + \Delta x q_x + \frac{\Delta x^2}{2}q_{xx} + \frac{\Delta x^3}{6}q_{xxx} + O(\Delta x^4) - \cancel{q}\right)$$
$$\left. - u\left(1 - \frac{\Delta t}{\Delta x}u\right)\left(\cancel{q} - \left(\cancel{q} - \Delta x q_x + \frac{\Delta x^2}{2}q_{xx} - \frac{\Delta x^3}{6}q_{xxx} + O(\Delta x^4)\right)\right)\right]$$

$$\Rightarrow \cancel{q - \frac{\Delta t}{\Delta x}\left[u(\right.}$$

$$\Rightarrow q - \frac{\Delta t}{\Delta x}u\left[\Delta t q_t - \frac{\Delta t^2}{2}q_{tt} + \frac{\Delta t^3}{6}q_{ttt} + O(\Delta t^4)\right]$$
$$- \frac{\Delta t}{2\Delta x}\left[\left(u + \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} + \frac{\Delta x^3}{6}u_{xxx} + O(\Delta x^4)\right)\left(1 - \frac{\Delta t}{\Delta x}u - \Delta t u_x - \frac{\Delta x}{2}\Delta t u_{xx} + O(\Delta^3)\right)\right.$$
$$\cdot\left(\Delta x q_x + \frac{\Delta x^2}{2}q_{xx} + \frac{\Delta x^3}{6}q_{xxx} + O(\Delta^4)\right)$$

$$-u\left(1+\frac{\Delta t}{\Delta x}u\right)\left(\Delta x\, q_x - \frac{\Delta x^2}{2} q_{xx} + \frac{\Delta x^3}{6} q_{xxx} + O(\Delta x^4)\right)\Bigg]$$

... this problem should be done on an algebraic calculation machine. With MMA one gets

$$-\Delta t\, \tau^n = (q_t + u(x) q_x)\,\Delta t \;+$$

$$\left(-\frac{1}{2}u(x) q_{tt} + \frac{1}{2}(u'(x) q_x + u(x) q_{xx})\right)\Delta x\,\Delta t \;+$$

$$\cancel{\left(-\frac{1}{24}u(x)\Delta x^3 q_{xxxx}\right.}$$

$$\left(\frac{1}{2}\left(\frac{1}{2}u''(x) q_x + \frac{1}{2}u'(x) q_{xx}\right) + \frac{1}{6}u(x) q_{xxx}\right)\Delta t\,\Delta x^2 + O(\Delta x^3)$$

$$+ \left(\frac{1}{2}q_{tt} + \frac{1}{2}(-2u(x)u'(x) q_x - u^2 q_{xx})\right)\Delta t^2 + \;+$$

$$\frac{1}{2}(-u$$

One has to show that the $O(\Delta t)$ terms vanish that the $O(\Delta x)$ terms vanish & that the $O(\Delta x \Delta t)$, $O(\Delta x^2)$ & $O(\Delta t^2)$ terms <u>don't</u> vanish. Using the Mathematica output this can be shown but I'm not sure what I'll gain by working through this ...

(9.2) The Lax–Wendroff approach consists of taylor series expansion at t_{n+1} & ~~[illegible]~~) use of the differential equation to evaluate the time derivatives

For the variable coefficient ~~[illegible]~~ color equation $q_t + u(x) q_x = 0$

$q_t = -u(x) q_x$

From Section 6.1

$$q(x, t_{n+1}) = q(x,t_n) + \Delta t\, q_t(x,t_n) + \tfrac{1}{2}\Delta t^2 q_{tt}(x,t_n) + O(\Delta t^3)$$

From $q_t = -u(x) q_x$ &

we have $q_{tt} = -u(x) q_{tx} = -u(x) \frac{\partial}{\partial x}(-u(x) q_x) = -u(x)\left[-u'(x) q_x - u(x) q_{xx}\right]$

$$= u(x) u'(x) q_x + u^2(x) q_{xx}$$

& this gives:

$$q(x,t_{n+1}) \hat{=} q(x,t_n) + \Delta t\,(-u(x) q_x(x,t_n)) + \tfrac{1}{2}\Delta t^2 \left(u(x)u'(x) q_x + u^2(x) q_{xx}\right) -$$

Using central finite difference approximations to the x derivatives gives:

$$Q_i^{n+1} = Q_i^n + -u(x_i)\frac{\Delta t}{\Delta x}\left(\frac{Q_{i+1}^n - Q_{i-1}^n}{2}\right) + \frac{1}{2}\Delta t^2 \left[u(x_i)\frac{(u(x_{i+1}) - u(x_{i-1}))}{2\Delta x}\frac{(Q_{i+1}^n - Q_{i-1}^n)}{2\Delta x} + u(x_i)^2 \frac{(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n)}{\Delta x^2}\right]$$

So

$$Q_i^{n+1} = Q_i^n - \frac{1}{2}\left(\frac{u_i \Delta t}{\Delta x}\right)(Q_{i+1}^n - Q_{i-1}^n) + \frac{\Delta t^2}{2\Delta x^2} u_i \left[\frac{(u_{i+1} - u_{i-1})}{2}\frac{(Q_{i+1}^n - Q_{i-1}^n)}{2} + u_i \frac{(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n)}{\Delta x^2}\right.$$

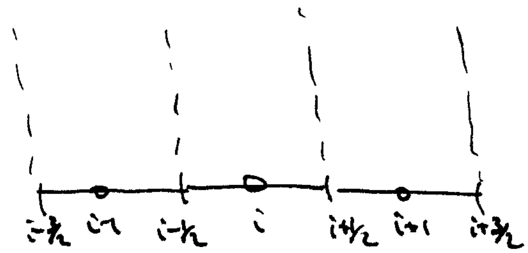
This is in contrast to eq 9.18 w/ eq 9.17

To compare these two methods I will compute ~~the~~ the ~~[illegible]~~ numerical approximation to each & see how each compares.

Because I can't think of an exact solution to test against (I didn't spend much time...) I will ~~test~~ compare each method against a much further refined grid than ~~[illegible]~~ either method predicts with. In addition showing the method is second order accurate is an exercise of Mathematica

(9.3) Eq 9.41 is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u_{i+1/2} Q_i^n - u_{i-1/2} Q_{i-1}^n\right)$$



Eq. 9.35 is

$$W_{i-1/2} = Q_i - Q_{i-1}$$

$$s_{i-1/2} = u_{i-1/2}$$

The method w/ second order corrections would then be

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(A^+ \Delta Q_{i+1/2} + A^- \Delta Q_{i-1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$$

w/ $\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{m} |s^p_{i-1/2}|\left(1 - \frac{\Delta t}{\Delta x}|s^p_{i-1/2}|\right)\tilde{W}^p_{i-1/2}$

with the expressions for $W_{i-1/2}$ + $s_{i-1/2}$ given above + Assuming that $u_{i-1/2} > 0$ one gets (with the 1D adv eq)

$$\tilde{F}_{i-1/2} = \cancel{\sum_{p=1}^{m}} \frac{1}{2} u_{i-1/2}\left(1 - \frac{\Delta t}{\Delta x} u_{i-1/2}\right)\left(Q_i - Q_{i-1}\right)$$

so Godunov's method written ~~in terms of~~ with second-order correction terms becomes:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u_{i+1/2} Q_i^n - u_{i-1/2} Q_{i-1}^n\right) -$$

$$- \frac{\Delta t}{\Delta x}\left(\frac{1}{2} u_{i+1/2}\left(1 - \frac{\Delta t}{\Delta x} u_{i+1/2}\right)(Q_{i+1} - Q_i) - \frac{1}{2} u_{i-1/2}\left(1 - \frac{\Delta t}{\Delta x} u_{i-1/2}\right)(Q_i - Q_{i-1})\right)$$

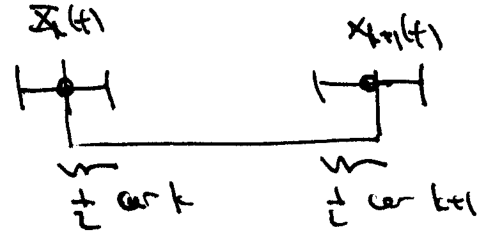
Then the local truncation error for this expression is given by

$$\tau^n = \frac{1}{\Delta t}\left[N(q^n) - q^{n+1}\right]$$ expanding w/ mathematica gives

$$= \dots$$

(9.4) Eq 9.32 is

$$q_k(t) = \frac{1}{X_{k+1}(t) - X_k(t)}$$



$\therefore$ In the distance $X_{k+1}(t) - X_k(t)$ we have a total of 1 ~~car~~ car length. $\therefore$ the density of cars in this interval is

$$q_k(t) = \frac{1}{X_{k+1} - X_k}$$

(9.5) $q_t + (u(x)q)_x = 0 \qquad q(x,0) = 1$

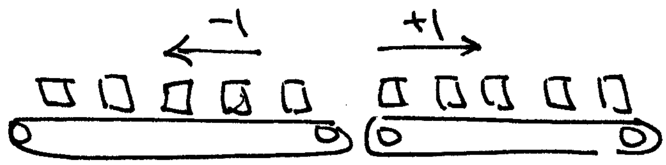
w/ $u(x) = \begin{cases} -1 & x < 0 \\ +1 & x > 0 \end{cases}$

For the cell to the left of the origin + to the right of the origin $(x=0)$ we have

$q_t - q_x = 0 \qquad x < 0 \qquad$ i.e. in cell $i-1 \qquad$ wave moving to the left i.e. $q(x,t) = q(x+t)$

$q_t + q_x = 0 \qquad x > 0 \qquad$ i.e. in cell $i \qquad$ wave moving to the right i.e. $q(x,t) = q(x-t)$

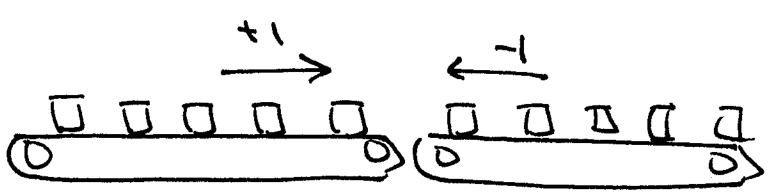
Thus we seem to have an expansion fan centered at $x=0$ + in terms of ~~the~~ the conveyor belt model we have



Since I know from later chapters that this situation corresponds to a transonic rarefaction, the methods presented here won't work very well, a special fix is needed.

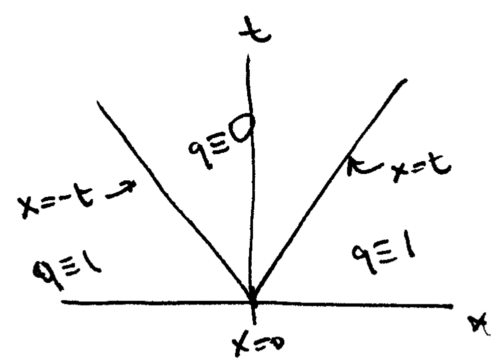
If $u(x) = \begin{cases} +1 & x < 0 \\ -1 & x > 0 \end{cases}$ the situation is reversed

& we have a pile up of density at the origin, resulting in two outward propagating shock waves. In terms of the conveyor-belt interpretation



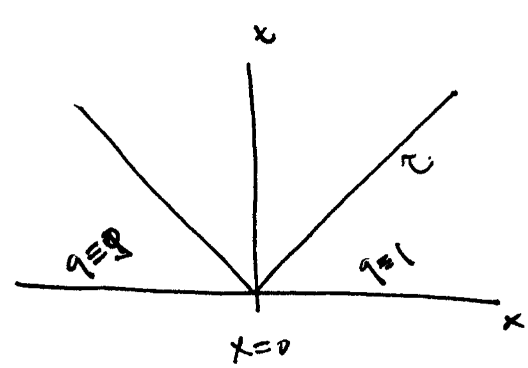
Analytically the solution to the 1st problem is

$$q(x,t) = \begin{cases} 1 & x < -t \\ 0 & -t < x < t \\ 1 & x > t \end{cases}$$

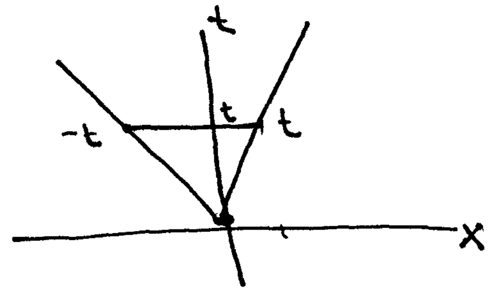


& the solution to the 2nd problem is

$$q(x,t) = \Big\{$$



$$1 \cdot \frac{1 \cdot \Delta t + 1 \cdot 1\, \Delta t}{\cancel{2\Delta t}\ 2\Delta t} = 1 = ?$$



I am not sure what the value of the state in the middle would be. I expect the density to increase but to what value?

(9.6) Section 9.5.2 deals w/ $q_t + (u(x)q)_x = 0$

giving

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u_{i+1/2}Q_i^n - u_{i-1/2}Q_{i-1}^n\right)$$

which is a 1st order update method. To develope a second order scheme we apply a unlimited "flux correction" giving.

$$\bar{Q}_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$$

with $\tilde{F}_{i-1/2} = \frac{1}{2}|s_{i-1/2}|\left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}|\right)\tilde{W}_{i-1/2}$

Using $W_{i-1/2} = Q_i - Q_{i-1}$

+ $s_{i-1/2} = u_{i-1/2}$ + Assuming that $u_j > 0 \quad \forall j$

Gives Assuming no limiter $\tilde{W}_{i-1/2} = W_{i-1/2}$

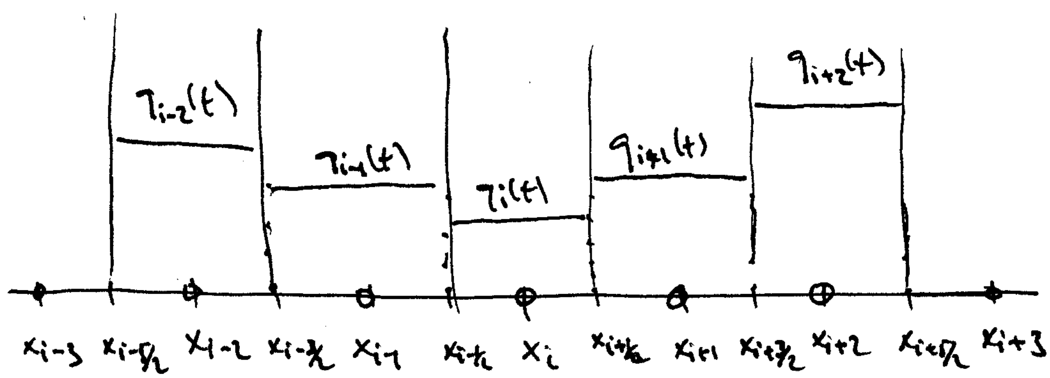
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u_{i+1/2}Q_i^n - u_{i-1/2}Q_{i-1}^n\right) - \frac{\Delta t}{\Delta x}\cdot$$

$$\left[\frac{1}{2}u_{i+1/2}\left(1 - \frac{\Delta t}{\Delta x}u_{i+1/2}\right)(Q_{i+1} - Q_i) - \frac{1}{2}u_{i-1/2}\left(1 - \frac{\Delta t}{\Delta x}u_{i-1/2}\right)(Q_i - Q_{i-1})\right]$$

Now

$\tau^n \equiv \frac{1}{\Delta t}\left[\mathcal{N}(q^n) - q^{n+1}\right]$ + using Mathematica to expand the every component gives ...

(10.1) ENO method uses $W_{i-j}, W_{i-j+1}, W_{i-j+2}, \dots, W_{i-j+s}$ to interpolate the value of q in cell C_i. The interpolation polynomial is based on j's between $1 + s$, we extend this idea. We select the value of $j \in [1,s]$ $\forall\, i \ni$ the interpolation through $W_{i-j}, \dots, W_{i-j+s}$ has the least oscillation over all possible choices $j = 1, 2, \dots, s$. Here least ~~oscill~~ oscillation is defined as the interpolating polynomial with the smaller divided difference.



$$W_i = W(x_{i+1/2}) = \int_{x_{1/2}}^{x_{i+1/2}} q(\xi, t)\, d\xi = \Delta x \sum_{j=1}^{i} \bar{q}_j(t)$$

Then using $W_{i-j}, \dots, W_{i-j+s}$ for $j \in [1,s]$ As true pointwise values of ~~W~~ W at ~~x~~ $x_{i-j+1/2}$ we can compute the ~~divided~~ newton interpolating polynomial through each set of

$W_{i-j}, \dots, W_{i-j+s}$ for $j \in [1,s]$ + take the one with the smallest oscillation

Then with j choosen (By the above)

$$p_i(x) = w(x) + O(\Delta x^{s+1}) \quad \text{in cell } C_i$$

so $p_i'(x) = q(x,t) + O(\Delta x^{s})$ " " "

then at the left + right cell interfaces

$$Q_i^L = p_i'(x_{i-1/2}) \quad + \quad Q_i^R = p_i'(x_{i+1/2})$$

In this problem $s=2$ so we have W_i $\forall i$ then $j \in [1,2]$ would have us select the 2nd order polynomial ~~from the following~~ interpolating the following pts

$j=1$ ~~from~~ W_{i-1}, W_i, W_{i+1}

$j=2$ W_{i-2}, W_{i-1}, W_i

? I think j should go from 0 + not 1. If so then we compare 3 polynomials for the smoothest in cell C_i

$j=0$ W_i, W_{i+1}, W_{i+2} used as fn values

$j=1$ W_{i-1}, W_i, W_{i+1} "

$j=2$ W_{i-2}, W_{i-1}, W_i "

But following the hint in the text should make this irrelevant.

Consider as $p_i^{(1)}(x)$ the linear polynomial passing through W_{i-1} & W_i

So
$$p_i^{(1)}(x) = W_{i-1} + \frac{(W_i - W_{i-1})}{(\Delta x \cancel{/2})}(x - \cancel{x_{i+1/2}} \; x_{i-1/2})$$

$$p_i^{(1)}(x) = W_{i-1} + \frac{(W_i - W_{i-1})}{\Delta x}(x - x_{i-1/2})$$

Next compute (the newton divided differences)

$$W[x_{i-3/2}, x_{i-1/2}, x_{i+1/2}] = \frac{W[x_{i-1/2}, x_{i+1/2}] - W[x_{i-3/2}, x_{i-1/2}]}{2\Delta x} = \cdots$$

$$\& \; W[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}] = \frac{W[x_{i+1/2}, x_{i+3/2}] - W[x_{i-1/2}, x_{i+1/2}]}{2\Delta x} = \cdots$$

& select for $p_i^{(2)}(x)$ the divided difference that is smaller in mag.

So
$$p_i^{(2)}(x) = \min\left(\left|W[x_{i-3/2}, x_{i-1/2}, x_{i+1/2}]\right|, \left|W[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]\right|\right)(x - \hat{x})(x - \hat{x})$$
$$+ \frac{(W_i - W_{i-1})}{\Delta x}(x - x_{i-1/2}) + W_{i-1}$$

w/ $\hat{x}$ & $\tilde{x}$ depending on which of $w[x_{i-3/2}, x_{i-1/2}, x_{i+1/2}]$ or $w[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]$ we choose.

If we pick $w[x_{i-3/2}, x_{i-1/2}, x_{i+1/2}]$ then

$$\hat{x} = x_{i-1/2} \quad \& \quad \tilde{x} = x_{i+1/2}$$

or if we pick $w[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]$ then

$$\hat{x} = x_{i-1/2} \quad \& \quad \hat{x} = x_{i+1/2}$$

So

$$\frac{dp_i^{(2)}(x)}{dx} = \min(|w[x_{i-3/2}, x_{i-1/2}, \dots |, | \dots |)(x-\tilde{x}) + \min(| \dots |, | \dots |)(x-\hat{x}) + \cancel{}$$

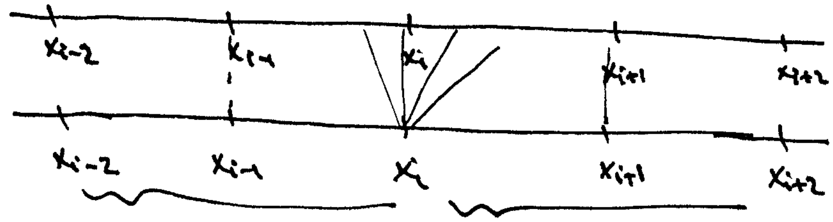
$$\cancel{x_i} \; w[x_{i+1/2}, x_i]$$

Then want to know

$$\frac{dp_1^{(2)}(x_i)}{dx} = \quad \dots$$

I don't see how this is related to the ~~xxxx~~ MinMod limiter ... should revisit ...

(10.2)



$\tilde{q}(x_{i-1},t^n) =$ $\quad$ $\tilde{q}(x_{i+1},t^n) =$

$Q^n_{i-1} + \frac{\sigma^n_{i-1}}{2}(x - x_{i-1})$ $\quad$ $Q^n_{i+1} + \frac{\sigma^n_{i+1}}{2}(x - x_{i+1})$

Then the 2nd order Nessyahu-Taylor scheme is given by eq 10.24 w/ $\tilde{q}^n$ Given in the cells ~~x_{i-1}~~ C_{i-1} & C_{i+1} as above. Thus to derive the method we must evaluate

$$\frac{1}{2\Delta x}\int_{x_{i-1}}^{x_{i+1}} \tilde{q}^n(x,t^n)\,dx = \frac{1}{2\Delta x}\left[\int_{x_{i-1}}^{x_i}\left(Q^n_{i-1} + \frac{\sigma^n_{i-1}}{2}(x-x_{i-1})\right)dx + \int_{x_i}^{x_{i+1}}\left(Q^n_{i+1} + \frac{\sigma^n_{i+1}}{2}(x-x_{i+1})\right)dx\right]$$

$$= \frac{1}{2\Delta x}\left[Q^n_{i-1}(\Delta x) + \frac{\sigma^n_{i-1}}{2}\frac{(x-x_{i-1})^2}{2}\Bigg|_{x_{i-1}}^{x_i} + Q^n_{i+1}\Delta x + \frac{\sigma^n_{i+1}}{2}\frac{(x-x_{i+1})^2}{2}\Bigg|_{x_i}^{x_{i+1}}\right]$$

$$= \frac{1}{2}(Q^n_{i-1} + Q^n_{i+1}) + \frac{1}{8\Delta x}\sigma^n_{i-1}\Delta x^2 + \frac{1}{8\Delta x}\sigma^n_{i+1}(-\Delta x^2)$$

$$= \frac{1}{2}(Q^n_{i-1} + Q^n_{i+1}) + \frac{1}{8}\Delta x(\sigma^n_{i-1} - \sigma^n_{i+1})$$

eq 10.27 ✓

Now $\quad F_{i-1}^{n+1/2} \triangleq f(\tilde{q}^{n}(x_{i-1}, t^{n+1/2}))$

$$\tilde{q}^{n}(x_{i-1}, t^{n+1/2}) \cong \tilde{q}^{n}(x_{i-1}, t^{n}) + \frac{\Delta t}{2}\, \tilde{q}^{n}_{t}(x_{i-1}, t^{n}) + O(\Delta t^{2})$$

By conservation law $\quad q_t = -f(q)_x$

so

$$\tilde{q}^{n}(x_{i-1}, t^{n+1/2}) \cong Q^{n}_{i-1} + -\frac{\Delta t}{2}\frac{\partial f}{\partial x}(Q^{n}_{i-1}) \qquad \text{eq} \quad 10.28 \checkmark$$

Then $\quad F_{i-1}^{n+1/2} = f(\;\text{"}\;) = f(Q^{n}_{i-1} - \frac{\Delta t}{2}\phi^{n}_{i-1})$

w/ $\quad \phi^{n}_{i-1} \cong \frac{\partial f}{\partial x}(Q^{n}_{i-1}) = \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} = f'(Q^{n}_{i-1})\,\sigma^{n}_{i-1}$

(11.1) $q_t + f(q)_x = 0 \qquad q(x,0) = \overset{\circ}{q}(x)$

Following the hint in the text, from eq 11.11, the solution to this equation is given by $q(x,t) = \overset{\circ}{q}(\xi)$ w/ ξ the solution to

$$x = \xi + f'(\overset{\circ}{q}(\xi))\,t$$

so $q(x,t) = \overset{\circ}{q}(\xi(x,t))$

$$\frac{\partial q(x,t)}{\partial x} = \frac{\partial \overset{\circ}{q}}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} \qquad \text{w/} \qquad \frac{\partial x}{\partial \xi} = 1 + f''(\overset{\circ}{q}(\xi))\,\overset{\circ}{q}'(\xi)\,t$$

so
$$\frac{\partial q(x,t)}{\partial x} = \frac{\partial \overset{\circ}{q}/\partial \xi}{1 + f''(\overset{\circ}{q}(\xi))\,\overset{\circ}{q}'(\xi)\,t}$$

Now $\frac{\partial q}{\partial x}$ will become infinite at a time T iff

$$1 + f''(\overset{\circ}{q}(\xi))\,\overset{\circ}{q}'(\xi)\,T = 0 \qquad + \text{ Again } \xi \text{ satisfies}$$
$$x \doteq \xi + f'(\overset{\circ}{q}(\xi))\,T$$

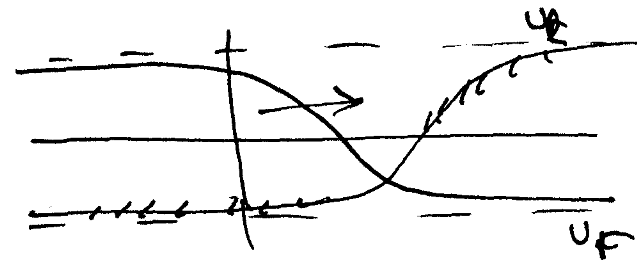
or
$$T = \frac{-1}{f''(\overset{\circ}{q}(\xi))\,\overset{\circ}{q}'(\xi)}$$

to compute the earliest time when this characteristic solution breaks down ∴

$$T_b = \mathrm{Min}\left(\frac{-1}{f''(\overset{\circ}{q}(\xi))\,\overset{\circ}{q}'(\xi)}\right) = -\mathrm{Max}\left(\frac{1}{f''(\overset{\circ}{q}(\xi))\,\overset{\circ}{q}'(\xi)}\right)$$

$$\therefore T_b = \frac{-1}{\min_{\xi}\left(f''(\overset{\circ}{q}(\xi))\,\overset{\circ}{q}'(\xi)\right)} \quad \text{or eq 11.55} \ \checkmark$$

(11.2) $u_t + u u_x = \epsilon u_{xx}$



let $u^\epsilon(x,t) = w^\epsilon(x-st)$ then

$u^\epsilon_t = -s w_\xi$ & $u^\epsilon_x = w^\epsilon_\xi$ etc. Putting this into the above gives

$$-s w_\xi + w w_\xi = \epsilon w_{\xi\xi} \quad \text{w/ limiting behavior} \quad \begin{aligned} w(+\infty) &= u_r \\ w(-\infty) &= u_e \end{aligned}$$

$$\Rightarrow \frac{d}{d\xi}\left(-sw + \frac{w^2}{2}\right) = \epsilon w_{\xi\xi} \quad . \quad \text{integrating once gives}$$

$$-sw + \frac{w^2}{2} = \epsilon w_\xi + C$$

~~[illegible]~~ to evaluate the constant C, & s, Take limits as $\xi \to \pm\infty$

$\xi \to +\infty$

$$-s u_r + \frac{u_r^2}{2} = 0 + C$$

$\xi \to -\infty$

$$-s u_e + \frac{u_e^2}{2} = C$$

Subtracting these two expressions gives

$$-s(u_r-u_\ell)+\frac{1}{2}(u_r^2-u_\ell^2)=0$$

$$\Rightarrow \quad s=\frac{1}{2}(u_r+u_\ell)\ , \quad \text{then} \quad c=-\frac{1}{2}(u_r+u_\ell)u_r+\frac{u_r^2}{2}=-\frac{u_r u_\ell}{2}$$

So P.E now becomes

$$-sw+\frac{w^2}{2}=tw_\xi-\frac{u_r u_\ell}{2} \quad \Rightarrow \quad w_\xi=\frac{1}{t}\left(\frac{w^2}{2}-sw+\frac{u_r u_\ell}{2}\right)$$

$$\Rightarrow \quad w_\xi=\frac{1}{2t}\left(w^2-(u_\ell+u_r)w+u_r u_\ell\right)$$

$$=\frac{1}{2t}(w-u_\ell)(w-u_r)$$

So

$$\frac{dw}{(w-w_\ell)(w-u_r)}=\frac{d\xi}{2t} \qquad \text{partial fractions expansion}$$

$$\parallel$$

$$\frac{A}{w-u_\ell}+\frac{B}{w-u_r}=\frac{d\xi}{2t}$$

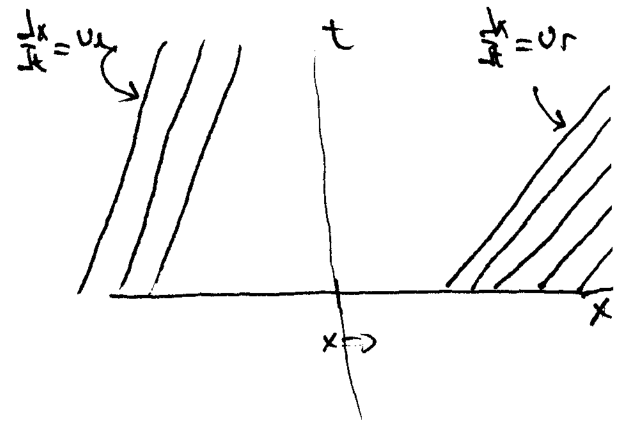
w/ $$A=\left.\frac{(w-u_\ell)}{(w-u_\ell)(w-u_r)}\right|_{w=u_\ell}=\frac{1}{u_\ell-u_r}$$

$$\& \quad B = \left.\frac{1}{w - U_l}\right|_{w=U_r} = \frac{1}{U_r - U_l}$$

~~So~~ Now, from the characteristic structure of this problem, for

~~$\frac{d\xi}{dt}$~~ $\cong$ $x \ll -1$ we have

$$\frac{dx}{dt} \cong +U_l \quad \& \quad \frac{dx}{dt} = +U_r$$



If $U_l < U_r$ then the characteristics don't intersect at all. Thus the information from $(x \ll -1)$ can never reach that at $x \gg +1$ since the speed is ~~so~~ ~~much~~ less, then that at $x \gg +1$. See the above diagram. Thus if $U_l < U_r$ we will have an expansion fan like phenomenon. For a travelling standing wave we ~~also~~ require $U_l > U_r$. Thus we can write our ODE now as (remembering that $U_r < w < U_l$)

$$\left(\frac{1}{U_l - U_r}\right)\left(\frac{-1}{U_l - w}\right) + \frac{-1}{(U_l - U_r)}\frac{1}{(w - U_r)} = \frac{d\xi}{2\epsilon}$$

$$\rightarrow \quad \frac{+1}{U_l - w} + \frac{1}{w - U_r} = -\frac{(U_l - U_r)}{2\epsilon}\, d\xi$$

$$-\ln(u_l - w) + \ln(w - u_r) = -\frac{(u_l - u_r)}{2\epsilon}(\xi) + C_2$$

$$\Rightarrow \quad \ln\left(\frac{w-u_r}{u_l-w}\right) = -\frac{(u_l-u_r)}{2\epsilon}\xi + C_2$$

$$= -\frac{(u_l-u_r)}{2\epsilon}(\xi - \xi_0) \qquad \text{Change constant } C_2 \text{ into } \xi_0$$

to solve $\ln\left(\frac{w-u_r}{u_l-w}\right) = x$ we perform.

$$\frac{w-u_r}{u_l-w} = e^x$$

$$w - u_r = (u_l - w)e^x$$

$$(1+e^x)w = u_r + u_l e^x$$

$$w = \frac{u_r + u_l e^x}{1+e^x} = \frac{u_r + (u_l - u_r + u_r)e^x}{1 + (2-1)e^x}$$

$$= \frac{u_r + u_r e^x + (u_l - u_r)e^x}{1+e^x} = u_r + (u_l-u_r)\frac{e^x}{1+e^x}$$

$$= \cancel{u_r + \frac{(u_l-u_r)}{2}\left(\frac{e^x + e^x}{1+e^x}\right)} = u_r + \frac{(u_l-u_r)}{2}\left[\frac{2e^x}{1+e^x}\right]$$

~~Multiply by $e^{x/2}$~~

$$= U_r + \frac{(U_\ell - U_r)}{2}\left[\frac{1+e^x + (e^x - 1)}{1+e^x}\right]$$

$$= U_r + \frac{(U_\ell - U_r)}{2}\left[1 + \frac{e^x - 1}{e^x + 1}\right]$$

$$= U_r + \frac{(U_\ell - U_r)}{2}\left[1 + \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}}\right]$$

$$= U_r + \frac{(U_\ell - U_r)}{2}\left[1 + \tanh(x/2)\right]$$

So in our problem

$$w(\xi) = U_r + \frac{(U_\ell - U_r)}{2}\left[1 + \tanh\left(-\frac{(U_\ell - U_r)}{4\epsilon}(\xi - \xi_0)\right)\right] \qquad \text{eq } 11.86 \checkmark$$

ξ_0 is determined by the initial conditions

(11.3) Eq 11.21 is $$s = \frac{f(q_r) - f(q_\ell)}{q_r - q_\ell}$$

let $q_r = q_\ell + (q_r - q_\ell)$ & expand $f(q_r)$ about q_ℓ in a taylor series

$$f(q_r) = f(q_\ell) + f'(q_\ell)(q_r - q_\ell) + \frac{f''(q_\ell)}{2}(q_r - q_\ell)^2 + O(|q_r - q_\ell|^3) \quad *$$

Now let

$q_\ell = q_r + (q_\ell - q_r)$ & taylor expand $f(q_\ell)$ about q_r

$$f(q_\ell) = f(q_r) + f'(q_r)(q_\ell - q_r) + \frac{f''(q_r)}{2}(q_\ell - q_r)^2 + O(|q_\ell - q_r|^3) \quad *$$

Now subtract eq * from eq ** & ÷ by $(q_r - q_\ell)$ to get

$$\cancel{\frac{f(q_\ell) - f(q_r)}{}}$$

$$\frac{f(q_\ell) - f(q_r)}{q_r - q_\ell} \approx$$

$$\frac{f(q_r) + f'(q_r)(q_\ell - q_r) + O(|q_\ell - q_r|^2) - f(q_\ell) - f'(q_\ell)(q_r - q_\ell) + O(|\;|^2)}{q_r - q_\ell}$$

$$\Rightarrow \frac{F(q_\ell)-F(q_r)}{q_r-q_\ell} \doteq \frac{F(q_r)-F(q_\ell)}{q_r-q_\ell} + \frac{F'(q_r)(q_\ell-q_r) - F'(q_\ell)(q_r-q_\ell)}{q_r-q_\ell}$$

$$+ \frac{F''(q_r)(q_\ell-q_r)^2/2 - F''(q_\ell)(q_r-q_\ell)^2/2}{q_r-q_\ell} + O(|q_r-q_\ell|^2)$$

$$\Rightarrow -S = S + \cancel{F(q_r)} - F'(q_r) - F'(q_\ell) + \frac{F''(q_r)(-1)(q_r-q_\ell) - F''(q_\ell)(q_r-q_\ell)}{2} + O(|\ |^2)$$

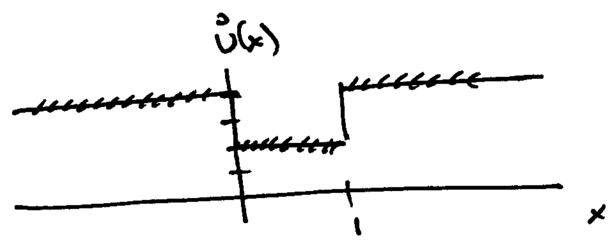
$$\Rightarrow -2S = -(F'(q_r) + F'(q_\ell)) + \quad ''$$

$$\Rightarrow S = \frac{F'(q_r)+F'(q_\ell)}{2} + -\frac{(F''(q_r)+F''(q_\ell))(q_r-q_\ell)}{4} + O(|\ |^2)$$

Now show $F''(q_r)+F''(q_\ell) = O(|q_r-q_\ell|)$ ... don't think this is true.

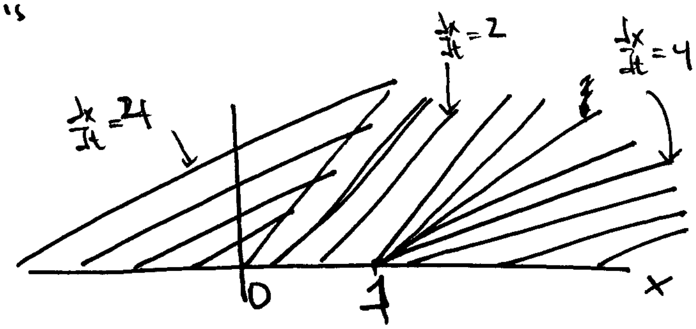
is result $O(\Delta^2)$ or $O(\Delta)$? What is wrong?

(11.6) $\mathring{u}(x) = \begin{cases} 2 & 0 < x < 1 \\ 4 & \text{else} \end{cases}$



The characteristic structure of Burger's eq is

$$\frac{dx}{dt} = u$$



Thus a shock forms at $x=0$

and a rarefaction fan at $x=1$.

Initially the shock wave ~~mo~~ moves at speed

$s = \frac{1}{2}(4+2) = 3$. This will intersect the "back" side of the

rarefaction fan at $x=1$ at a time T_c when

$$x_c = 3T_c \quad \& \quad x_c = 2T_c + 1 \quad \text{or}$$

$3T_c = 2T_c + 1 \to T_c = 1$ at $x_c = 3$. From this point onward

the shock speed is governed by

$$\dot{x}_s(t) = \frac{1}{2}(4 + q(x/t)) \quad w/ \quad q(x/t) \text{ the solution to Burger's eq}$$

in the rarefaction fan. In this case, that solution is

$$f'(q(x/t)) = \frac{x-1}{t} \quad \text{from eq } 11.27 \qquad \text{Now } f'(q) = \frac{d}{dq}\left(\tfrac{1}{2}q^2\right) = q$$

for Burger's eq

$$\| $$

$$q(x/t) = \frac{x-1}{t} \quad \text{"inside" the rarefaction fan}$$

So by method (a)

$$\dot{x}_s(t) = \frac{1}{2}\left(4 + \frac{x-1}{t}\right)$$

$$\Rightarrow \quad \dot{x}_s - \frac{1}{2t}x_s = 2 - \frac{1}{2t} \qquad \checkmark$$

mult by $e^{\int -\frac{d\tau}{2\tau}} = e^{-\frac{1}{2}\ln t} = \frac{1}{\sqrt{t}}$ to get

$$\underbrace{t^{-1/2}\dot{x}_s - \frac{1}{2}t^{-3/2}x_s} = 2t^{-1/2} - \frac{1}{2}t^{-3/2}$$

$$\frac{d}{dt}\left(t^{-1/2}x_s\right) = 2t^{-1/2} - \frac{1}{2}t^{-3/2}$$

$$t^{-1/2}x_s(t) = \frac{2t^{1/2}}{\frac{1}{2}} - \frac{1}{2}\frac{t^{-1/2}}{\left(-\frac{1}{2}\right)} + C_1$$

$$\Rightarrow \quad t^{-1/2}x_s(t) = 4t^{1/2} + t^{-1/2} + C_1$$

$$x_s(t) = 4t + 1 + C_1 t^{1/2}$$

Check: $\dot{x}_s(t) = 4 + 0 + \frac{1}{2}C_1 t^{-1/2}$

& ~~$\frac{x_s-1}{t}$~~ ~~and~~ $x_s - 1 = 4t + C_1 t^{1/2}$

So $\frac{x_s-1}{t} = 4 + C_1 t^{-1/2}$ thus

$$c_1 t^{-1/2} = \frac{x_s - 1}{t} - 4 \quad ✓$$

So $\dot{x}_s(t) = 4 + \frac{1}{2}\left(\frac{x_s-1}{t} - 4\right) = 2 + \frac{x_s - 1}{2t} = \frac{1}{2}\left(4 + \frac{x_s-1}{t}\right)$ ✓

Since $x_s(t=1) = 3$ we get

$$\begin{gathered} x_s(t=1) \\ \parallel \\ 4 + 1 + c_1(1) = 3 \Rightarrow c_1 = -2 \end{gathered}$$

So $x_s(t) = 4t + 1 - 2t^{1/2}$

Now by method (b) Don't quite see how to construct ~~this~~ the solution in this way

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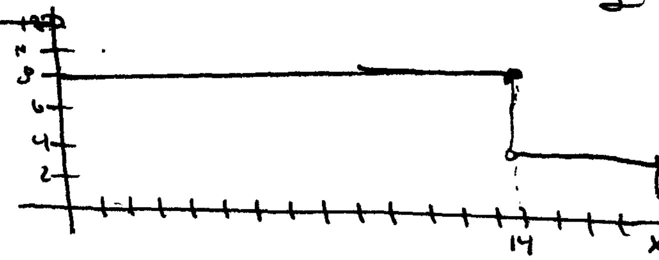
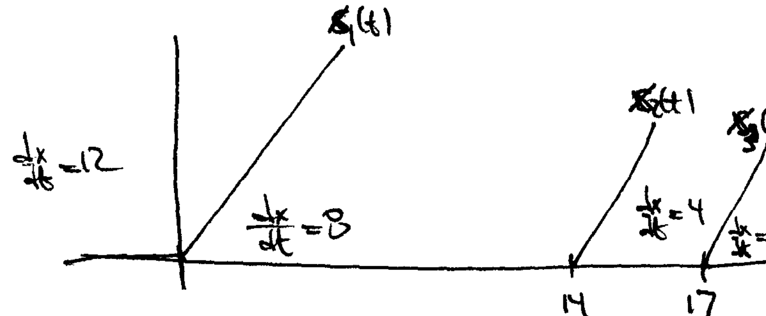
(11.7)

$$\overset{\circ}{u}(x) = \begin{cases} 12 & x<0 \\ 8 & 0<x<14 \\ 4 & 14<x<17 \\ 2 & 17<x \end{cases}$$



Then for Burgers eq has characteristic structure $\frac{dx}{dt} = u$

so the discontinuity at $x=0$ develops into a shock
at $x=14$ " " " "
at $x=18$ " " " "



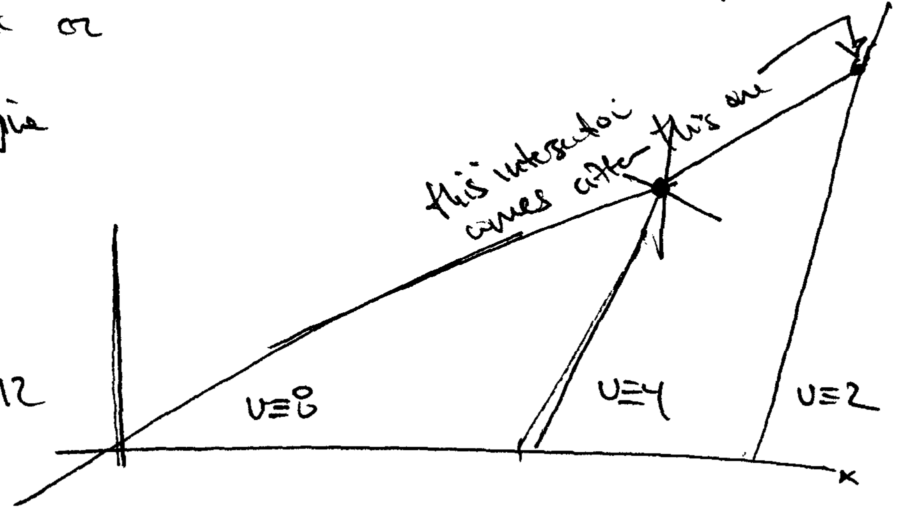
Then the speeds of each shock are

$s_i(t) = \frac{1}{2}(u_l+u_r)$ & they give

$s_1(t) = \frac{1}{2}(12+8) = 10$

$s_2(t) = \frac{1}{2}(8+4) = 6$

$s_3(t) = \frac{1}{2}(4+2) = 3$



Each shock will intersect & change its speed

so $x_1(t) = 10t$; $x_2(t) = 6t+14$; $x_3(t) = 2t+17$

The 1st intersection is at $\bar{X} = 10t = 6t+14$

$4t = 14 \Rightarrow t = 7$

+ then moves w/ speed $\hat{s}_2(t) = \frac{1}{2}(12+4) = 8$

Q: Does the shock from $x=14$ + $x=17$ intersect before $t=7$

$6t + 14 = 3t + 17$

$3t = 3 \Rightarrow t = 1$ Yes it does!

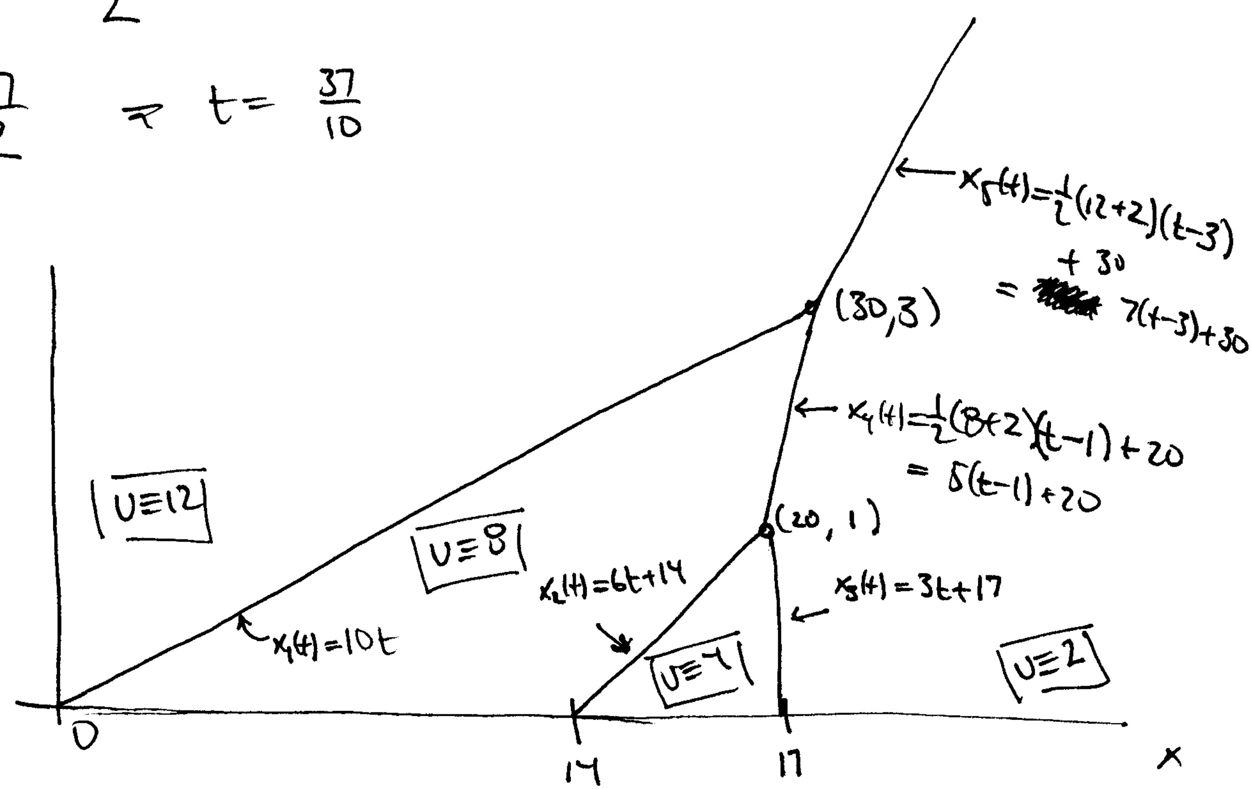
This will produce another shock at $x = 2(1) + 17 = 19$

$\hat{s}_4(t) = \frac{1}{2}(8+2) = 5$

This shock will intersect the one from $x=0$ at

$10t = 5t + \frac{37}{2}$

$\Rightarrow 5t = \frac{37}{2} \Rightarrow t = \frac{37}{10}$



Then shock $x_4(t)$ intersects shock at $x_1(t)$ when

$$10t = 5(t-1) + 20$$

$$5t = 15 \Rightarrow t = 3 \quad + \quad x = 30$$

See the previous diagram for a characterization of the entire flow field as a function of time.

11.9 Eq 11.4 $f(q) = qU(q)$ w/ conservation law

$$q_t + (qU(q))_x = 0 \quad *$$

An entropy function $\eta(\cdot)$ + entropy flux $\psi(\cdot)$ are scalar functions such that

$$\eta(q)_t + \psi(q)_x \le 0$$

w/ $\eta(\cdot)$ a convex fn of q w/ $\eta''(q) > 0 \ \forall q$.

Eq $*$ becomes.

mult by $2q$

$$q_t + U(q)q_x + qU'(q)q_x = 0$$

let $\eta(q) = q^2$ Then $q = \eta^{1/2}$ $U(q) = U(\eta^{1/2})$

So $\eta_t + \psi_x = 2qq_t +$

$$s(\eta(q_r) - \eta(q_l)) \ge \psi(q_r) - \psi(q_l)$$

... Not sure ... this should be easy, in fact by Godunov ~~[illegible]~~ a ~~[illegible]~~ all symmetric systems of which this is a special case should have a quadratic entropy fn $\eta(q) = q^2$ ...
But I couldn't get this to work...

$$TV_T(q) = \limsup_{\epsilon \to 0} \frac{1}{\epsilon}\int_0^T \int_{-\infty}^{+\infty} |q(x+\epsilon,t) - q(x,t)|\,dx\,dt$$

$$+ \limsup_{\epsilon \to 0} \frac{1}{\epsilon}\int_0^T \int_{-\infty}^{+\infty} |q(x,t+\epsilon) - q(x,t)|\,dx\,dt$$

If $q(x,t)$ is piecewise constant, then

$$TV_T(q) = \sum_{n=0}^{T/\Delta t} \sum_{i=-\infty}^{+\infty} \cancel{\Delta t}\,\Delta t\,|Q^n_{i+1} - Q^n_i| + \sum_{n=0}^{T/\Delta t} \sum_{i=-\infty}^{+\infty} \frac{1}{\cancel{\Delta t}}\Delta x \cancel{\Delta t}\,|Q^{n+1}_i - Q^n_i|$$

$$= \sum_{n=0}^{T/\Delta t} \sum_{i=-\infty}^{+\infty} \left[\Delta t\,|Q^n_{i+1} - Q^n_i| + \Delta x\,|Q^{n+1}_i - Q^n_i|\right] \quad \text{eq } 12.51$$

since $\epsilon \cong \Delta x$ or Δt

+ $|q(x+\epsilon,t) - q(x,t)| \cong |Q^n_{i+1} - Q^n_i|$

+ $|q(x,t+\epsilon) - q(x,t)| \cong |Q^{n+1}_i - Q^n_i|$

Then eq 12.51 becomes w/ $\|Q^n_*\|_1 = \Delta x \sum_{i=-\infty}^{+\infty} |Q^n_i|$

$$TV_T(q) = \sum_{n=0}^{T/\Delta t} \left[\Delta t\, TV(Q^n) + \|Q^{n+1} - Q^n\|_1\right] \quad \text{eq } 12.52$$

$$TV_T(Q^{\Delta t}) = \sum_{n=0}^{T/\Delta t} \left[TV(Q^n)\,\Delta t + \|Q^{n+1} - Q^n\|_1 \right]$$

since $\|Q^{n+1} - Q^n\|_1 \le \alpha \Delta t$ $\qquad$ $TV(Q^n) \le R$

$$TV_T(Q^{\Delta t}) \le \sum_{n=0}^{T/\Delta t} (\Delta t R + \alpha \Delta t) = (\Delta t R + \alpha \Delta t)\left(\frac{T}{\Delta t}\right) = (R+\alpha)\,T$$

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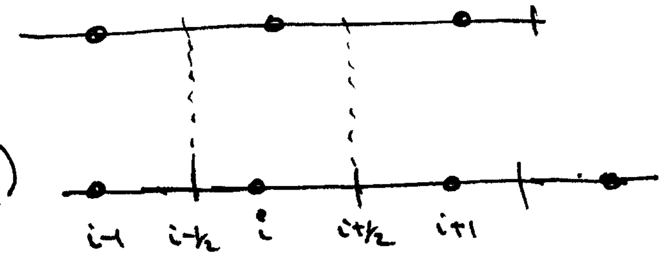
(12.1) Method 12.5



$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right)$$

$$A^+\Delta Q_{i-1/2} = f(Q_i) - f(Q^{\downarrow}_{i-1/2})$$

is Godunov's method

$$A^-\Delta Q_{i-1/2} = f(Q^{\downarrow}_{i-1/2}) - f(Q_{i-1})$$

(12.8) is $A^+\Delta Q_{i-1/2} = s^+_{i-1/2} W_{i-1/2}$ & $A^-\Delta Q_{i-1/2} = s^-_{i-1/2} W_{i-1/2}$ } if $Q^{\downarrow}_{i-1/2} = Q_{i-1}$ or Q_i

Then w/

$$A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2} = s^+_{i-1/2} W_{i-1/2} + s_{i+1/2} W_{i+1/2}$$

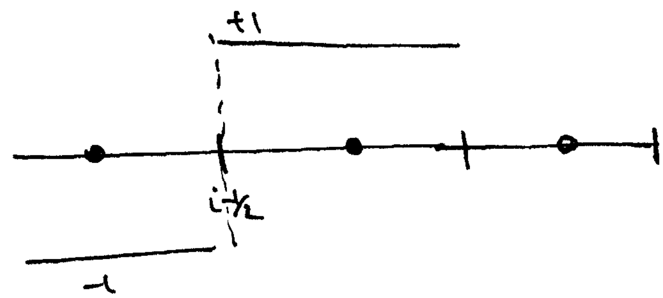
$$= \phi$$

Not sure how to do this problem?

(b) Assuming pt (a) is correct then the numerical flux is given by

$$F_{i-1/2} = \frac{1}{2}\left[f(Q_{i-1}) + f(Q_i) - a_{i-1/2}(Q_i - Q_{i-1})\right]$$

w/ $$a_{i-1/2} = \left|\frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}\right|$$



For Burger's equation $f(q) = \frac{1}{2}q^2$

So if $q_\ell = Q_{i-1} = -1$

& $q_r = Q_i = +1$

Then $f(q_\ell) = f(q_r) = 1$ & $a_{i-1/2} = 0$ so

$$F_{i-1/2} = \frac{1}{2}[1+1-0] = 1$$

Then $F_{i+1/2} = \frac{1}{2}[1+1-0] = 1$

$$\therefore \quad Q^{n+1} = Q^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right)$$

Thus $Q_i^{n+1} = Q_i^n$ $\forall i$ & the ~~shock~~ discontinuity is stationary.

The correct solution shall be a rarefaction fan.

12.2

12.12 is LLF w/

$$F_{i-1/2} = \frac{1}{2}\left[f(Q_{i-1}) + f(Q_i) - a_{i-1/2}(Q_i - Q_{i-1})\right]$$

w/ $a_{i-1/2} = \mathrm{Max}(|f'(q)|)$ $\forall$ q between Q_{i-1} + Q_i

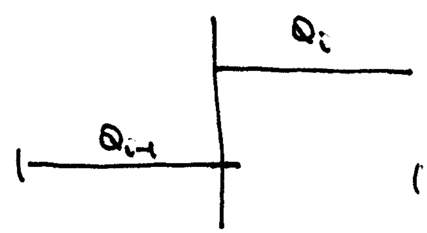
Osher's definition of an E-scheme is if

$$\mathrm{sgn}(Q_i - Q_{i-1})(F_{i-1/2} - f(q)) \le 0 \quad \forall\ q \text{ between } Q_{i-1} + Q_i$$

For the LLF method this becomes:

$$\mathrm{sgn}(Q_i - Q_{i-1})\left[\frac{1}{2}\left(f(Q_{i-1}) - f(q) + f(Q_i) - f(q) - a_{i-1/2}(Q_i - Q_{i-1}\right)\right] \overset{?}{\le} 0$$

Assume that $Q_i > Q_{i-1}$



Then $\mathrm{sgn}(Q_i - Q_{i-1}) = +1$

+ the above can be written

$$\frac{1}{2}\frac{\mathrm{sgn}(Q_i - Q_{i-1})}{(Q_i - Q_{i-1})}\left[\frac{f(Q_{i-1}) - f(q)}{Q_i - Q_{i-1}} + \frac{f(Q_i) - f(q)}{Q_i - Q_{i-1}} - \mathrm{Max}(|f'(q)|)\right] \overset{?}{\le} 0$$

If f is monotonic then we are done since one of the quotient terms will be negative + the other will be negative by subtraction ... Not sure ...

1

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(12.3) An E-scheme is one

$$\operatorname{sgn}(Q_i - Q_{i-1})(F_{i-1/2} - f(q)) \le 0 \qquad q \text{ between } Q_i \text{ + } Q_{i-1}$$

Then by Harten's thm involving a sufficient theorem for a scheme to be TV diminishing we look for schemes of the form

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n(Q_i^n - Q_{i-1}^n) + D_i^n(Q_{i+1}^n - Q_i^n)$$

or numerical scheme is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

$$= Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - f(q) - F_{i-1/2}^n + f(q))$$

$$= Q_i^n - \underbrace{\frac{\Delta t}{\Delta x}\frac{(F_{i+1/2}^n - f(q))}{(Q_{i+1}^n - Q_i^n)}}_{D_i^n}(Q_{i+1}^n - Q_i^n) + \underbrace{\frac{\Delta t}{\Delta x}\frac{(F_{i-1/2}^n - f(q))}{(Q_i^n - Q_{i-1}^n)}}_{-C_{i-1}^n}(Q_i^n - Q_{i-1}^n)$$

~~Defin~~

Then checking each condition of Harten's

$$C_{i-1}^n = -\frac{\Delta t}{\Delta x}\frac{(F_{i-1/2}^n - f(q))}{(Q_i^n - Q_{i-1}^n)} \geq 0 \quad ✓$$

$$D_i^n = -\frac{\Delta t}{\Delta x}(F^n \qquad) \geq 0 \quad ✓$$

$$C_i^n + D_i^n = -\frac{\Delta t}{\Delta x}\left[+\frac{(F_{i+1/2} - f(q))}{Q_{i+1} - Q_i} + \frac{F_{i+1/2} - f(q)}{Q_{i+1} - Q_i}\right] = -2\frac{\Delta t}{\Delta x}\left[\frac{F_{i+1/2} - f(q)}{Q_{i+1} - Q_i}\right]$$

Assuming $\frac{F_{i+1/2} - f(q)}{Q_{i+1} - Q_i}$ can be Bounded i.e $\exists M >$

$|f'(q)| \leq M$ ~~thus~~ ~~expansion~~ then for sufficiently small $\frac{\Delta t}{\Delta x}$

on ~~E~~ Osher E-scheme is TVD

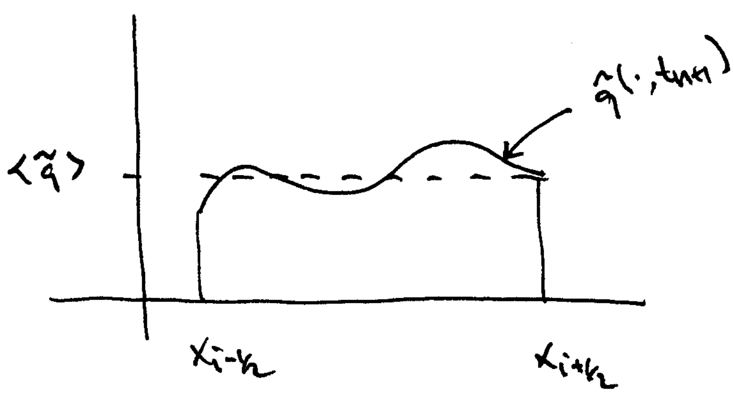
(12.4) 12.41 is

$$\eta\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^{n}(x,t_{n+1})\,dx\right) \leq \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta(\tilde{q}(x,t_{n+1}))\,dx$$

which is true $\forall$ convex ~~$q(\cdot,t_{n+1})$~~ $\eta(\cdot)$

i.e. ~~$q''(\cdot,t_{n+1}) > 0$~~

$\eta''(\cdot) > 0$



let $\eta = x^3$ ~~then~~ with $x_{i-1/2} = -1$ + $x_{i+1/2} = +1$ with $\hat{q} \equiv 1$

$\Rightarrow \Delta x = 2$

$$\left(\frac{1}{2}\int_{-1}^{+1} dx\right)^3 \overset{?}{\leq} \frac{1}{2}\int_{-1}^{+1} 1\,dx = 1$$

$\left(\frac{2}{2}\right)^3 \leq 1$ O.K. not a good counter example try $\tilde{q} = x$ wont work either

try $\tilde{q} = x^2$

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}\,dx = \frac{1}{2}\int_{-1}^{+1} x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta(\hat{q}(x,t_{n+1}))\,dx = \frac{1}{2}\int_{-1}^{+1} x^6\,dx = \frac{x^7}{7}\Big|_0^1 = \frac{1}{7}$$

Check $\left(\frac{1}{3}\right)^3 \overset{?}{\leq} \frac{1}{7}$ yes.

... ran out of time ...

$h_t + (uh)_x = 0$ 1st eq in 13.5

2nd eq in 13.5 is

$h_t u + h u_t + h_x u^2 + 2 u u_x h + g h h_x = 0$ putting in 1st eq from 13.5

gives

$$u(-u_x h - \cancel{h_x u}) + h u_t + \cancel{u^2 h_x} + 2 h u u_x + g h h_x = 0$$

$$h u_t + h u u_x + g h h_x = 0$$

$\Rightarrow u_t + (\tfrac{1}{2} u^2)_x + g h_x = 0$ eq 13.6 ✓

$$\text{w/ } f(q) = \begin{bmatrix} q^2 \\ \dfrac{(q^2)^2}{q^1} + \dfrac{1}{2} g (q^1)^2 \end{bmatrix}$$

$$f'(q) = \begin{bmatrix} 0 & 1 \\ -\dfrac{(q^2)^2}{(q^1)^2} + g(q^1) & \dfrac{2q^2}{q^1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix}$$ eq 13.8 ✓

$$\begin{vmatrix} -\lambda & 1 \\ -u^2+gh & 2u-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \; -\lambda(2u-\lambda) + u^2 - gh = 0$$

$$\lambda^2 - 2u\lambda + (u^2 - gh) = 0$$

$$\lambda = \frac{2u \pm \sqrt{4u^2 - 4(1)(u^2-gh)}}{2} = \frac{2u \pm \sqrt{\cancel{4u^2} - \cancel{4u^2} + 4gh}}{2}$$

$$= u \pm \sqrt{gh} \qquad \text{eq } 13.9 \; ✓$$

Then eivectors are for $\lambda' = u - \sqrt{gh}$

$$\begin{bmatrix} -u+\sqrt{gh} & 1 \\ -u^2+gh & 2u-u+\sqrt{gh} \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -u+\sqrt{gh} & 1 \\ -u^2+gh & u+\sqrt{gh} \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -u+\sqrt{gh} & 1 \\ (-u+\sqrt{gh})(u+\sqrt{gh}) & u+\sqrt{gh} \end{bmatrix} \cdots = 0$$

$\therefore$ if $r^1 = \begin{pmatrix} r \\ s \end{pmatrix}$

$$(-u+\sqrt{gh})r + s = 0 \Rightarrow s = (u-\sqrt{gh})r$$

so $r^1 = \begin{pmatrix} 1 \\ u-\sqrt{gh} \end{pmatrix}$

sim. $r^2 = \begin{pmatrix} 1 \\ u+\sqrt{gh} \end{pmatrix}$ is obtained.

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$s = \frac{h_* u_* - hu}{h_* - h}$ into 2nd of eq 13.15 gives

$\frac{(h_* u_* - hu)^2}{h_* - h} = h_* u_*^2 - hu^2 + \frac{1}{2} g (h_*^2 - h^2)$ one eq. can be solved for $u = u(h; h_*, u_*)$

$$\Rightarrow \cancel{h_*^2 u_*^2} - 2h_* u_* h u + \cancel{h^2 u^2} = \cancel{h_*^2 u_*^2} - h_* h u^2 + \frac{1}{2} g h_* (h_*^2 - h^2)$$
$$- h h_* u_*^2 + \cancel{h^2 u^2} - \frac{1}{2} g h (h_*^2 - h^2)$$

$$\Rightarrow + h_* h u^2 - 2h_* u_* h u + h h_* u_*^2 + \frac{gh}{2}(h_*^2 - h^2) - \frac{g h_*}{2}(h_*^2 - h^2)$$

$\div\, h_* h$ to get

$$+ u^2 - 2u_* u + u_*^2 + \underbrace{\frac{g}{2h_*}(h_*^2 - h^2) - \frac{g}{2h}(h_*^2 - h^2)}_{\frac{g}{2}(h_*^2 - h^2)\left[\frac{1}{h_*} - \frac{1}{h}\right]} = 0$$

$\therefore$

$$u^2 - 2u_* u + u_*^2 + \frac{g}{2}(h_*^2 - h^2)\frac{(h - h_*)}{h_* h} = 0$$

$$\Rightarrow \quad u^2 - 2u_* u + u_*^2 + \frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h - h_*) = 0$$ eq 13.17 ✓

$$\Rightarrow \quad u = \frac{2u_* \pm \sqrt{4u_*^2 - 4\left[u_*^2 + \frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h-h_*)\right]}}{2}$$

$$\Rightarrow \quad u = u_* \pm \sqrt{\cancel{u_*^2} - \cancel{u_*^2} - \frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h-h_*)}$$

$$\Rightarrow \quad u = u_* \pm \sqrt{\frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h_* - h)}$$ ✓

~~h_*~~ $h > h_*$ so $\frac{h_*}{h} < 1$ + $\frac{h}{h_*} > 1$

$\therefore \frac{h_*}{h} - \frac{h}{h_*} < 0$ + since $(h_* - h) < 0$ the object under the

sq. root is ~~neg~~ positive ✓

let $h = h_* + \alpha$

Then

$hu = \cancel{(h_* + \alpha)(u_* \pm \sqrt{\frac{g}{2}(\frac{h_*}{h} - \frac{h}{h_*})(h_* - h)}}\,()$

so $$u = u_* \pm \sqrt{\frac{g}{2}\left(\frac{h_*^2 - h^2}{h h_*}\right)(h_* - h)}$$

w/ [redacted] $h = h_* + \alpha$

$$h^2 = h_*^2 + 2h_*\alpha + \alpha^2 \Rightarrow h_*^2 - h^2 = -2h_*\alpha - \alpha^2$$

$$u = u_* \pm \sqrt{\frac{g}{2}\frac{(2h_*\alpha + \alpha^2)}{h_*(h_*+\alpha)}(\alpha)}$$

~~hu_*~~ ± Then

$$hu = (h_* + \alpha)\left(u_* \pm \sqrt{\frac{g}{2}\frac{(2h_*\alpha + \alpha^2)}{h_*(h_*+\alpha)}\alpha}\right)$$

$$= h_*u_* \cancel{\pm h_* \sqrt{}} + \alpha u_* \cancel{\pm} \alpha \sqrt{\frac{g}{2}\frac{(2h_*\alpha + \alpha^2)}{\cancel{h_*(h_*+\alpha)}}\alpha}$$

$$= h_*u_* + \alpha\left[u_*\right] \pm \underbrace{(h_*+\alpha)\sqrt{\frac{g}{2}\frac{(2h_*\alpha + \alpha^2)\alpha}{h_*(h_*+\alpha)}}}$$

$$\underbrace{\sqrt{\frac{g}{2}\frac{(2h_*\alpha + \alpha^2)\alpha}{h_*}(h_*+\alpha)}}$$

$$|\alpha|\sqrt{\frac{g}{2}\frac{(h_*+\alpha)(2h_*+\alpha)}{h_*}}$$

$$\therefore \quad hu = h_* u_* + \alpha \left[u_* \pm \underbrace{\sqrt{\frac{gh_*}{2}\left(1+\frac{\alpha}{h_*}\right)\left(2+\frac{\alpha}{h_*}\right)}}_{\sqrt{gh_*\left(1+\frac{\alpha}{h_*}\right)\left(1+\frac{\alpha}{2h_*}\right)}} \right]$$

eq 13.18 ✓

Thus
$$q = \begin{pmatrix} h \\ hu \end{pmatrix} = \begin{pmatrix} h_* + \alpha \\ h_* u_* + \alpha\left(u_* \pm \sqrt{gh_*\left(1+\frac{\alpha}{h_*}\right)\left(1+\frac{\alpha}{2h_*}\right)}\right) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} h_* \\ h_* u_* \end{pmatrix}}_{q_*} + \alpha \begin{pmatrix} 1 \\ u_* \pm \underbrace{\sqrt{gh_*\left(1+\frac{\alpha}{h_*}\right)\left(1+\frac{\alpha}{2h_*}\right)}}_{\sqrt{gh_*\left(1+O(\alpha)\right)}} \end{pmatrix}$$

$\alpha \ll 1$

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$$h_m u_m = h_r u_r + \alpha\left[u_r + \sqrt{g h_r\left(1+\frac{\alpha}{h_r}\right)\left(1+\frac{\alpha}{2h_r}\right)}\,\right]$$

+ $h_m = h_r + \alpha$ $\rightarrow$ $\alpha = h_m - h_r$ in the above gives:

$$h_m u_m = h_r u_r + (h_m - h_r)\left[u_r + \sqrt{g h_r\left(1+\frac{(h_m-h_r)}{h_r}\right)\left(1+\frac{h_m-h_r}{2h_r}\right)}\,\right]$$

$$= \cancel{h_r u_r} + h_m u_r + h_m\sqrt{g h_r\left(1+\frac{(h_m-h_r)}{h_r}\right)\left(1+\frac{h_m-h_r}{2h_r}\right)}$$

$$- \cancel{h_r u_r} \qquad - h_r\sqrt{g h_r\left(1+\frac{(h_m-h_r)}{h_r}\right)\left(1+\frac{h_m-h_r}{2h_r}\right)}$$

The sqrt term becomes:

$$g h_r\left(\cancel{1}+\frac{h_m}{h_r}-\cancel{1}\right)\left(1+\frac{1}{2}\frac{h_m}{h_r}-\frac{1}{2}\right)$$

$$= g h_m\left(\frac{1}{2}+\frac{1}{2}\frac{h_m}{h_r}\right) = \frac{g h_m}{2}\left(1+\frac{h_m}{h_r}\right) \quad \text{so we get}$$

so we get

$$h_m u_m = h_m u_r + h_m\sqrt{\frac{g h_m}{2}\left(1+\frac{h_m}{h_r}\right)} - h_r\sqrt{\frac{g h_m}{2}\left(1+\frac{h_m}{h_r}\right)}$$

$$\Rightarrow \quad u_m = u_r + \sqrt{\frac{g h_m}{2}\left(1+\frac{h_m}{h_r}\right)} - \frac{h_r}{h_m}\sqrt{}$$

$$\text{or} \quad u_m = u_r + \left(1-\frac{h_r}{h_m}\right)\sqrt{\frac{g h_m}{2}\left(1+\frac{h_m}{h_r}\right)}$$

$$\Rightarrow \quad v_m = v_r + \left(\overset{h_m}{\cancel{h_m}} - h_r\right)\sqrt{\frac{g}{2h_m}\left(1 + \frac{h_m}{h_r}\right)}$$

$$\Rightarrow \quad v_m = v_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)} \qquad \text{eq } 13.19$$

For the left shock...

$$h_m u_m = h_L u_L + \alpha\left[u_L - \sqrt{g h_L\left(1+\frac{\alpha}{h_L}\right)\left(1+\frac{\alpha}{2h_L}\right)}\right]$$

$$\text{\& } h_m = h_L + \alpha \Rightarrow \alpha = h_m - h_L$$

Using the minus sign to link the left state to the middle state

$$\Rightarrow h_m u_m = h_L u_L + (h_m - h_L)\left[u_L - \sqrt{\underbrace{g h_L\left(1+\frac{h_m-h_L}{h_L}\right)\left(1+\frac{(h_m-h_L)}{2h_L}\right)}}\right]$$

$$\underbrace{g h_L\left(\cancel{1}+\frac{h_m}{h_L}-\cancel{1}\right)\left(1+\frac{h_m}{2h_L}-\frac{1}{2}\right)}$$

$$\underbrace{g h_m\left(\frac{1}{2}+\frac{h_m}{2h_L}\right)}$$

$$\frac{g h_m}{2}\left(1+\frac{h_m}{h_L}\right)$$

$$\Rightarrow h_m u_m = h_L u_L + (h_m - h_L)\left[u_L - \sqrt{\frac{g h_m}{2}\left(1+\frac{h_m}{h_L}\right)}\right]$$

$$= \cancel{h_L u_L} + h_m u_L - \cancel{h_L u_L} - (h_m - h_L)\sqrt{\frac{g h_m}{2}\left(1+\frac{h_m}{h_L}\right)}$$

$\div h_m$ gives

$$u_m = u_L - \left(1 - \frac{h_L}{h_m}\right)\sqrt{\frac{g h_m}{2}\left(1+\frac{h_m}{h_L}\right)}$$

$$u_m = u_L - (h_m - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h_m}+\frac{1}{h_L}\right)} \qquad \text{eq } 13.20 \checkmark$$

$$\tilde{q}'(\xi) = \alpha(\xi)\, r^{p}(q(\xi))$$

w/ $\alpha = 1$ the 1 rarefaction wave curves are given by the solution to the following ODE's w/ $\tilde{q}^1 = \tilde{h}$ + $\tilde{q}^2 = (\tilde{hu})$

$$\tilde{q}'(\xi) = \begin{bmatrix} 1 \\ U - \sqrt{gh} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\tilde{q}^2}{\tilde{q}^1} - \sqrt{g\tilde{q}^1} \end{bmatrix}$$

So in components we get:

$$\frac{d(\tilde{q}^1)}{d\xi} = 1 \qquad + \qquad \frac{d}{d\xi}(\tilde{q}^2) = \frac{\tilde{q}^2}{\tilde{q}^1} - \sqrt{g\tilde{q}^1}$$

$\Rightarrow$ $\tilde{q}^1(\xi) = \xi$ then the 2nd equation becomes

$$\frac{d}{d\xi}(\tilde{q}^2) = \frac{\tilde{q}^2}{\xi} - \sqrt{g\xi}$$

So that

$$\frac{d}{d\xi}(\tilde{q}^2) - \frac{\tilde{q}^2}{\xi} = -\sqrt{g\xi}$$

An integrating factor for this D.E. is

$$e^{\int -\frac{1}{\xi} d\xi} = e^{-\ln \xi} = \frac{1}{\xi}.$$ Multiplying by this one obtains

$$\underbrace{+\frac{1}{\xi}\frac{d}{d\xi}(\tilde{q}^2) - \frac{1}{\xi^2}\tilde{q}^2} = -\frac{\sqrt{g\xi}}{\xi}$$

$$\frac{d}{d\xi}\left(\frac{1}{\xi}\tilde{q}^2\right) = -\frac{\sqrt{g}}{\sqrt{\xi}}$$

$$\frac{\tilde{q}^2}{\xi} = -\frac{\sqrt{g}\,\xi^{1/2}}{1/2} + C$$

$$\Rightarrow \quad \tilde{q}^2 = -2\sqrt{g\xi}\cdot\xi + C\cdot\xi$$

$$\Rightarrow \quad \tilde{q}^2 = -2\sqrt{g}\,\xi^{3/2} + C\cdot\xi$$

If $\xi = h = \tilde{q}^1$ one can impose the initial condition that

$$\tilde{q}^2(h_*) = h_* u_*$$

$$\|$$

$$-2\sqrt{g}\,h_*^{3/2} + C h_* = h_* u_*$$

$\Rightarrow \quad C = u_* + 2\sqrt{gh_*}$ to give

$$\tilde{q}^2(\xi) = -2\sqrt{g}\,\xi^{3/2} + (u_* + 2\sqrt{gh_*})\xi$$

$$= -2\sqrt{g}\,\xi^{1/2}\cdot\xi + (u_* + 2\sqrt{gh_*})\xi$$

Thus

$$\tilde{q}^2(\xi) = U_* \xi + 2\xi\left(\sqrt{gh^*} - \sqrt{g\xi}\right) \qquad \text{eq 13.30 ✓}$$

For r^2 the integral curves for the shallow water eqs are w/ parametrization $\alpha(\xi) \equiv 1$

$$\frac{d\tilde{q}(\xi)}{d\xi} = r^2(\tilde{q}(\xi)) = \begin{bmatrix} 1 \\ \hat{U} + \sqrt{g\hat{h}} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\tilde{q}^2(\xi)}{\tilde{q}^1(\xi)} + \sqrt{g\tilde{q}^1(\xi)} \end{bmatrix}$$

So In components we obtain:

$$\frac{d\tilde{q}^1(\xi)}{d\xi} = 1 \quad +$$

$$\frac{d\tilde{q}^2(\xi)}{d\xi} = \frac{\tilde{q}^2(\xi)}{\tilde{q}^1(\xi)} + \sqrt{g\tilde{q}^1(\xi)}$$

The 1st eq is as before $\tilde{q}^1(\xi) = \xi$ while the 2nd has ξ^{-1} as an integrating factor

$$\Rightarrow \cancel{\xi\frac{d\tilde{q}^2(\xi)}{d\xi} - \tilde{q}^2(\xi) = \xi\sqrt{g\xi}} \quad \text{or}$$

$$\frac{1}{\xi}\frac{d\tilde{q}^2(\xi)}{d\xi} - \frac{1}{\xi^2}\tilde{q}^2(\xi) = +\frac{\sqrt{g\xi}}{\xi}$$

$$\Rightarrow \quad \frac{d}{d\xi}\left(\frac{1}{\xi}\tilde{q}^2(\xi)\right) = \frac{\sqrt{g\xi}}{\xi}$$

$$\tilde{q}^2(\xi) = \left(\frac{\sqrt{g}\,\xi^{-1/2+1}}{1/2} + C\right)\xi$$

$$\Rightarrow \quad \tilde{q}^2(\xi) = 2\sqrt{g}\,\xi^{1/2}\cdot\xi + C\cdot\xi$$

w/ I.C. of $\tilde{q}^2(\xi = h_*) = h_* u_*$ we get

$$h_* u_* = 2\sqrt{g}\,h_*^{1/2}h_* + C h_* \quad \Rightarrow \quad C = u_* - 2\sqrt{g}\,h_*^{1/2}$$

So $\quad \tilde{q}^2(\xi) = 2\sqrt{g}\,\xi^{1/2}\cdot\xi + u_*\xi - 2\sqrt{g}\,h_*^{1/2}\xi$

$$= u_*\xi - 2\xi\left(\sqrt{gh_*} - \sqrt{g\xi}\right)$$

So associating $\tilde{q}^2(\xi)$ w/ $u\cdot h$ & $\tilde{q}^1(\xi) = \xi$ w/ h

$$uh = u_* h - 2h\left(\sqrt{gh_*} - \sqrt{gh}\right)$$

$$\Rightarrow \quad u = u_* - 2\left(\sqrt{gh_*} - \sqrt{gh}\right) \qquad \text{eq } 13.33 \ \checkmark$$

$$\lambda^1 = u - \sqrt{gh} \qquad \cancel{\lambda^2 = u + \sqrt{gh}}$$

$$= \frac{q^2}{q^1} - \sqrt{g q^1}$$

$$\nabla \lambda^1 = \left(\frac{\partial}{\partial q^1}(\lambda^1), \frac{\partial}{\partial q^2}(\lambda^1) \right) = \left(\frac{-q^2}{(q^1)^2} - \sqrt{g}\,(q^1)^{1/2-1}\,\frac{1}{2}, \frac{1}{q^1} \right)$$

$$= \begin{bmatrix} -\frac{q^2}{(q^1)^2} - \frac{1}{2}\sqrt{\frac{g}{q^1}} \\ \frac{1}{q^1} \end{bmatrix} \qquad \text{eq } 13.49$$

$$r^1 = \begin{bmatrix} 1 \\ \frac{q^2}{q^1} - \sqrt{g q^1} \end{bmatrix}$$

$$\nabla \lambda^1 \cdot r^1 = \cancel{-\frac{q^2}{(q^1)^2}} - \frac{1}{2}\sqrt{\frac{g}{q^1}} + \cancel{\frac{q^2}{(q^1)^2}} - \sqrt{\frac{g}{q^1}} = -\frac{3}{2}\sqrt{\frac{g}{q^1}} \quad \text{eq}$$

13.50 ✓

So then eq 13.48 becomes

$$\tilde{q}'(\xi) = \begin{bmatrix} \frac{d\hat{q}^1(\xi)}{d\xi} \\ \frac{d\hat{q}^2(\xi)}{d\xi} \end{bmatrix} = -\frac{2}{3}\sqrt{\frac{q^1}{g}} \begin{bmatrix} 1 \\ \frac{q^2}{q^1} - \sqrt{g q^1} \end{bmatrix} \qquad \text{eq } 13.51 ✓$$

$$\tilde{h}'(\xi) = -\frac{2}{3}\sqrt{\frac{\tilde{h}(\xi)}{g}} = -\frac{2}{3}\frac{\sqrt{\tilde{h}(\xi)}}{\sqrt{g}}$$

$$\Rightarrow \quad \frac{d\tilde{h}}{\sqrt{\tilde{h}}} = -\frac{2}{3}\frac{1}{\sqrt{g}}\,d\xi$$

$$\Rightarrow \quad \tilde{h}^{-1/2}\,d\tilde{h} = -\frac{2}{3}\frac{1}{\sqrt{g}}\,d\xi$$

$$\frac{\tilde{h}^{1/2}}{1/2} = -\frac{2}{3}\frac{1}{\sqrt{g}}\xi + C$$

$$\tilde{h} = \left(C - \frac{1}{3}\frac{1}{\sqrt{g}}\xi\right)^2 = \frac{1}{9g}(A-\xi)^2 \qquad \text{eq } 13.52$$

$\tilde{h} = h_l$ at $\xi = u_l - \sqrt{gh_l}$

So $h_l = \frac{1}{9g}\left(A - u_l + \sqrt{gh_l}\right)^2 \Rightarrow \pm 3\sqrt{gh_l} = A - u_l + \sqrt{gh_l}$

$$\Rightarrow \quad A = u_l + \begin{Bmatrix} 2 \\ -2 \end{Bmatrix}\sqrt{gh_l}$$

But also $\tilde{h} = h_r$ at $\xi = u_r - \sqrt{gh_r}$ thus gives

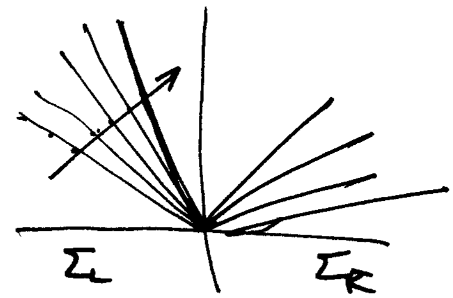
$$h_R = \frac{1}{9g}\left(A - u_r + \sqrt{gh_r}\right) \Rightarrow A = u_r + \begin{Bmatrix} 2 \\ -2 \end{Bmatrix}\sqrt{gh_r}$$

But A can only be 1 constant & from the discussion of Riemann invariants for 1 waves

$u_\ell + 2\sqrt{gh_\ell} = u_r + 2\sqrt{gh_r}$ if [redacted] $(h_\ell, h_\ell u_\ell)$ + $(h_R, h_R u_R)$ fall

on the same rarefaction curve as they must. Thus the contact

$$A = u_\ell + 2\sqrt{gh_\ell} = u_r + 2\sqrt{gh_r}$$



$u_m = u_L + 2(\sqrt{gh_L} - \sqrt{gh_m})$ ~~on~~ r^1 Riemann curve

$u_m = u_R - 2(\sqrt{gh_R} - \sqrt{gh_m})$ r^2 Riemann curve

~~$u_L = u_r \Rightarrow$~~ $u_L + 2(\sqrt{gh_L} - \sqrt{gh_m}) = u_r - 2(\sqrt{gh_R} - \sqrt{gh_m})$

$$\frac{u_L - u_r}{2} + \sqrt{gh_L} + \sqrt{gh_R} = 2\sqrt{gh_m}$$

$$\Rightarrow \quad h_m = \frac{1}{4g}\left(\sqrt{gh_L} + \sqrt{gh_R} + \frac{u_L - u_r}{2}\right)^2$$

$$= \frac{1}{16g}\left(u_L - u_r + 2(\sqrt{gh_L} + \sqrt{gh_r})\right)^2$$

With $u_L = -.5$ $u_r = .5$

$h_\ell = 1$ $h_r = 1$ we get

$$h_m = \frac{1}{16g}\left(-.5 - .5 + 4\sqrt{g}\right)^2 = \frac{1}{16g}\left(-1 + 4\sqrt{g}\right)^2 = \dots$$

Py 293 LeVeque

$$pV = nRT$$

R universal gas constant $= 8.314\ J/mol \cdot K$.

n = # of moles of given gas.

$$\rho = \frac{n}{V} RT \qquad [\rho] = \frac{kg}{m^3}$$

m = mass of given gas $= n \cdot W$ w/ W = molecular weight of the gas

so $n = \frac{m}{W}$

so

$$p = \left(\frac{m}{W}\right)\frac{1}{V} RT = \left(\frac{m}{V}\right)\left(\frac{R}{W}\right)\cdot T = \rho\left(\frac{R}{W}\right)\cdot T$$

with $\rho = \frac{m}{V}$ + define $R_2 \equiv \frac{R}{W}$ another gas specific constant since it depends on the molecular weight of the gas.

$E_t + ((E+p)u)_x = 0$ w/ $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2$

As the polytropic ~~...~~ expression for an ideal gas.

we get

$$\frac{p_t}{\gamma-1} + \frac{1}{2}(\rho u^2)_t + \frac{\partial}{\partial x}\left[\left(\frac{\gamma}{\gamma-1}p + \frac{1}{2}\rho u^2\right)u\right] = 0$$

$$= \frac{p_t}{\gamma-1} + \frac{1}{2}(\rho u^2)_t + \frac{\partial}{\partial x}\left[\frac{\gamma}{\gamma-1}pu + \frac{1}{2}\rho u^3\right] = 0$$

$$\Rightarrow \frac{p_t}{\gamma-1} + \frac{1}{2}(\rho u^2)_t + \frac{\gamma}{\gamma-1}pu_x + \frac{\gamma}{\gamma-1}p_x u + \frac{1}{2}\rho u^2 u_x + \frac{1}{2}(\rho u^2)_x u = 0$$

$$\Rightarrow \frac{p_t}{\gamma-1} + \frac{1}{2}(\rho u)_t u + \frac{1}{2}(\rho u)u_t + \frac{\gamma}{\gamma-1}pu_x + \underbrace{\frac{\gamma}{\gamma-1}p_x u}_{= \, p_x u + \frac{1}{\gamma-1}p_x u} + \frac{1}{2}\rho u^2 u_x + \frac{1}{2}(\rho u^2)_x u = 0$$

combining $\frac{1}{2}u(\rho u)_t + p_x u + \frac{1}{2}u(\rho u^2)_x = \cancel{p_x u} + \frac{1}{2}p_x u$

from conservation of mass

So we have

~~$\gamma...$~~ $=$ $1-c = \frac{1}{2}$, $c = \frac{1}{2}$

$$\frac{p_t}{\gamma-1} + \frac{1}{2}(\rho u)u_t + \frac{\gamma}{\gamma-1}pu_x + \frac{1}{\gamma-1}p_x u + \frac{1}{2}p_x u + \frac{1}{2}\rho u^2 u_x = 0$$

But conservation of mass gives $u_t = -uu_x - \frac{1}{\rho}p_x$

$$\frac{p_t}{\gamma-1} - \cancel{\tfrac{1}{2}\rho u^2 u_x} - \cancel{\tfrac{1}{2} u \rho_x} + \frac{\gamma}{\gamma-1} p u_x + \frac{1}{\gamma-1} p_x u + \cancel{\tfrac{1}{2}\rho_x u} + \cancel{\tfrac{1}{2}\rho u^2 u_x} = 0$$

$$\Rightarrow \quad p_t + \gamma p u_x + p_x u = 0 \qquad \text{eq } 14.37 \ \checkmark$$

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0$$

eigen values:

$$\begin{vmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \gamma p & u-\lambda \end{vmatrix} = 0 \qquad \text{w/ } c^2 \equiv \frac{\gamma p}{\rho}$$

$(u-\lambda)\left[(u-\lambda)^2 - c^2\right] = 0 \quad \Rightarrow \quad \lambda = u$ is one eigenvalue

in addition $(\lambda-u)^2 = c^2$

$\Rightarrow \quad \lambda = u \pm c$ are two others. ordering them so

$\lambda^1 \leq \lambda^2 \leq \lambda^3$ we have $\lambda^1 = u-c$; $\lambda^2 = u$; $\lambda^3 = u+c$

Then $\vec{r}^1$ is given by the solution to

$$\begin{bmatrix} c & \rho & 0 \\ 0 & c & 1/\rho \\ 0 & \gamma p & c \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \begin{bmatrix} 1 & p/c & 0 \\ 0 & c & \gamma p \\ 0 & \gamma p & c \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} 1 & p/c & 0 \\ 0 & 1 & \gamma/cp \\ 0 & \gamma p & c \end{bmatrix} \;'' \;=\; ''$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & -\frac{p}{c}\cdot\frac{1}{cp} \\ 0 & 1 & \gamma/cp \\ 0 & 0 & c-\frac{\gamma p}{cp} \end{bmatrix} \;'' \;=\; ''$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & -\frac{1}{c^2} \\ 0 & 1 & \frac{1}{cp} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = 0$$

$$r = \frac{1}{c^2}t \qquad \text{so} \qquad r' = \begin{bmatrix} 1/c^2 \\ -\frac{1}{cp} \\ 1 \end{bmatrix} \cdot t$$

$$+ \; s = -\frac{1}{cp}t$$

For normalization to produce an $Z \equiv cp$ pick $t = -cp$

Then $$r' = \begin{bmatrix} -p/c \\ +1 \\ -cp \end{bmatrix}$$

For $\lambda^2 = 0$

$$\begin{bmatrix} 0 & p & 0 \\ 0 & 0 & 1/p \\ 0 & \gamma p & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \underline{0}$$

r is arbitrary & $t \equiv 0$ + $s \equiv 0$ so

$$r^2 = r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{pick } r \equiv 1 \quad \text{so} \quad r^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Then for $\lambda^3 = 0 + c$

$$\begin{bmatrix} -c & p & 0 \\ 0 & -c & 1/p \\ 0 & \gamma p & -c \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \underline{0}$$

$$\Rightarrow \begin{bmatrix} 1 & -p/c & 0 \\ 0 & 1 & -1/pc \\ 0 & \gamma p & -c \end{bmatrix} \; '' = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{p}{c}\frac{1}{pc} \\ 0 & 1 & -1/pc \\ 0 & 0 & -c + \frac{\gamma p}{pc} \end{bmatrix} \; '' = ''$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{c^2} \\ 0 & 1 & -1/pc \\ 0 & 0 & 0 \end{bmatrix} \; '' = '' \qquad \text{w/ } c^2/c = c$$

so $r = \frac{1}{c^2} \cdot t$ in component form: for r^3

+ $s = \frac{1}{\rho c} \cdot t$

So $r^3 = \begin{bmatrix} \frac{1}{c^2} \\ \frac{1}{\rho c} \\ 1 \end{bmatrix} \cdot t$ pick $t = \rho c$ then

$$r^3 = \begin{bmatrix} \rho/c \\ 1 \\ \rho c \end{bmatrix} \checkmark$$

$$\nabla \lambda^1 = \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial u}, \frac{\partial}{\partial p}\right) \lambda^1 =$$

$$c = \sqrt{\frac{\gamma p}{\rho}} \Rightarrow \frac{\partial c}{\partial \rho} = \frac{1}{2}\left(\frac{\gamma p}{\rho}\right)^{-1/2} \frac{\gamma p(-1)}{\rho^2} = \frac{1}{2} c^{-1} \cdot (-1) \frac{\gamma p}{\rho} \cdot \frac{1}{\rho} = \frac{1}{2} c^{-1}(-1) c^2 \cdot \frac{1}{\rho}$$

$$\frac{\partial c}{\partial p} = \frac{1}{2}\left(\frac{\gamma p}{\rho}\right)^{-\frac{1}{2}} \frac{\gamma}{\rho} = \frac{1}{2} c^{-2} \frac{\gamma}{\rho} = \frac{1}{2} c^{-2} \frac{\gamma p}{\rho} \cdot \frac{1}{p} = \frac{1}{2} \frac{c^{-1} \cdot c^2}{p}$$

so $\frac{\partial c}{\partial \rho} = -\frac{c}{2\rho}$ + $\frac{\partial c}{\partial u} = 0$

+ $\frac{\partial c}{\partial p} = \frac{c}{2p}$

so $\nabla \lambda^1 = \left(\frac{c}{2\rho}, 1, -\frac{c}{2p}\right)$ ✓

+ $\nabla \lambda^2 = \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial u}, \frac{\partial}{\partial p}\right) \lambda^2 = (0, 1, 0)$ ✓

+ $\nabla \lambda^3 = \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial u}, \frac{\partial}{\partial p}\right) \lambda^3 = \left(-\frac{c}{2\rho}, 1, \frac{c}{2p}\right)$ ✓

Then $\nabla \lambda^1 \circ r^1 = \frac{c}{2\rho} \cdot \left(\frac{-\rho}{c}\right) + 1 + \frac{-c}{2p} \cdot (-\rho c) = \frac{1}{2} + \frac{c^2 \rho}{2p}$

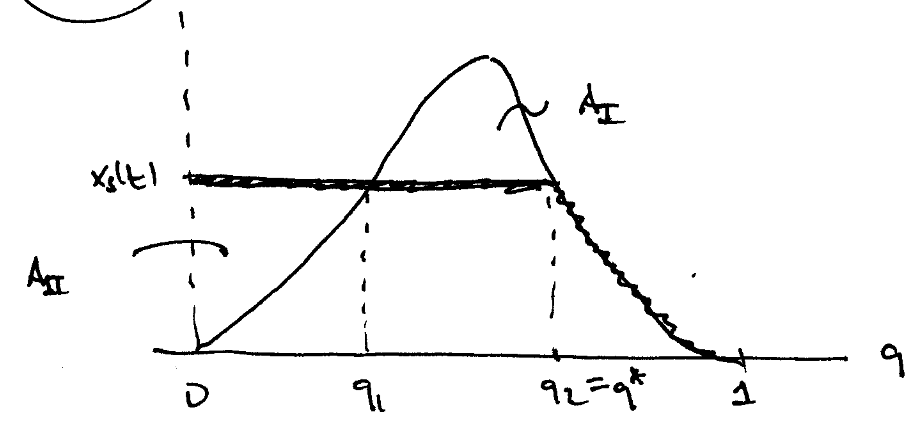
$= \frac{1}{2}(1 + \gamma)$ ✓

$$\nabla \lambda^2 \circ r^2 = 0$$

$$\nabla \lambda^3 \circ r^3 = -\frac{c}{2p} \cdot \frac{p}{c} + 1 + \frac{c}{2p} \cdot pc$$

$$= \frac{1}{2} + \frac{1}{2}\frac{p}{p} \cdot c^2 = \frac{1}{2}(1+r)$$

16.2 cont



Now ~~$A_I = \int_{q_1}^{q_2} tF'(q)\,dq = t(F(q_2$~~

$$A_I = \int_{q_1}^{q_2} \left(tF'(q) - x_s\right) dq = t\left(F(q_2) - F(q_1)\right) - x_s(q_2 - q_1)$$

$$+ \quad A_{II} = \int_{0}^{q_1} \left(x_s - tF'(q)\right) dq = x_s(q_1) - t\left(F(q_1) - F(0)\right)$$

$$= x_s q_1 - tF(q_1)$$

Setting the two areas equal gives

$$A_I = A_{II}$$

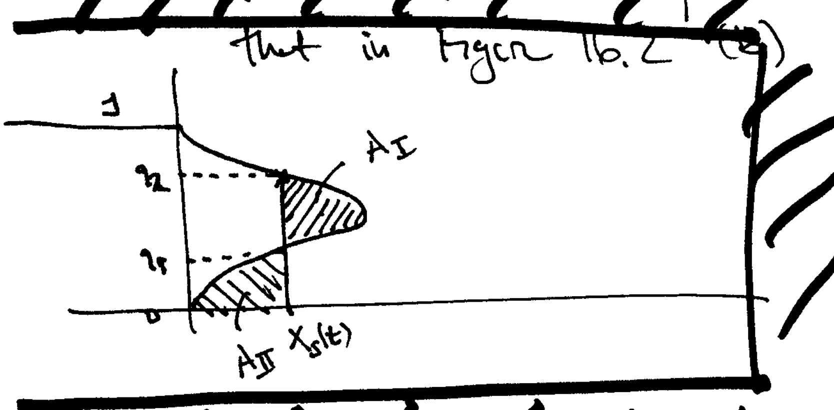
$$t\left(F(q_2) - \cancel{F(q_1)}\right) - x_s(q_2 - \cancel{q_1}) = \cancel{x_s q_1} - \cancel{tF(q_1)}$$

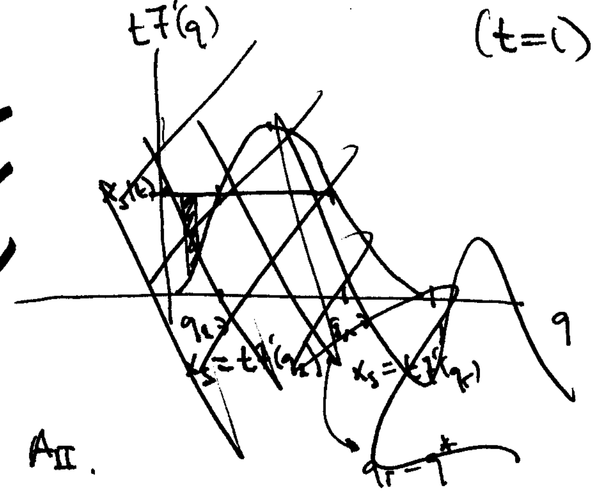
$$\Rightarrow \quad tF(q_2) - x_s q_2 = 0$$

$$\Rightarrow \quad x_s = \frac{tF(q_2)}{q_2} \quad \ast$$

But the point q_2 is the right most root of $tF'(q_2) = x_s$

The equal area rule then uses this information to construct an entropy satisfying shock by ~~drawing~~ plotting the profile $tf'(q)$ & drawing a vertical line representing the shock ~~which~~ which would cut off equal lobes. In this problem the profile looks like that in Figure 16.2 (c)





So A_I must have equal area to A_{II}.

Now

Michele,

Can you make this into a graphic?

& (there is another one in two pages)

eps file would be great

(John Wex)

$(f(q_r) - f(q_\ell))$

$f(q_\ell) - f(0)) \neq t(f$

A_{II} ~~be located at such a location~~

That

$$A_I = \int_0^{q_\ell} (x_s - tf'(q))\,dq$$

&

$$A_{II} = \int_{q_\ell}^{q_r} (tf'(q) - x_s)\,dq$$