Problem Solutions & Derivations for Finite Volume Methods for Hyperbolic Problems by Randall J. LeVeque

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Introduction

A Note on Notation

In these notes, I use the symbol $\Rightarrow$ to denote the results of elementary elimination matrices used to transform a given matrix into its reduced row echelon form. Thus when looking for the eigenvectors for a matrix like

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

rather than say, multiplying A on the left by

$$E_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

produces

$$E_{33}A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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we will use the much more compact notation

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Additional Notes And Derivations:

$w^1(x,t)$ satisfies the one-way wave equation (Page 2)

Given the acoustic equations

$$p_t + Ku_x = 0 \quad (1)$$

$$u_t + \frac{1}{\rho}p_x = 0 \quad (2)$$

where the constants ρ and K the density and bulk modulus of compressibility respectively. Define $w^1(x,t) = p + \rho c u$ with $c = \sqrt{\frac{K}{\rho}}$. Now consider

$$w_t^1 + cw_x^1 = p_t + \rho c u_t + cp_x + c^2 \rho u_x \quad (3)$$

$$= -Ku_t + \rho c \left(-\frac{1}{\rho} p_x \right) + cp_x + c^2 \rho u_x \quad (4)$$

$$= -Ku_x - c\rho_x + cp_x + Ku_x = 0 \quad (5)$$

Where from Eq. 3 to Eq. 4 we have used Eq. 1 to eliminate the time derivatives and between Eq. 4 and Eq. 5 we have used the definition of c.

Letting $w^2 = p - c\rho u$ and we obtain

$$w_t^2 - cw_x^2 = p_t - c\rho u_t - cp_x + c^2 \rho u_x \quad (6)$$

$$= -Ku_x + c \left(\frac{\rho}{\rho} p_x \right) - cp_x + c^2 \rho u_x \quad (7)$$

$$= -Ku_x + cp_x - cp_x + Ku_x \quad (8)$$

using many of the same substitutions as before.

Analytic Solution to the Linear Acoustic Equations by way of Characteristic Decomposition (Page 30-31)

Consider the matrix of right eigenvectors for the acoustic equation, given by

$$R = \begin{bmatrix} -Z_0 & Z_0 \\ 1 & 1 \end{bmatrix}. \quad (9)$$

Which has an inverse given by the standard trick for inverting 2x2 systems

$$R^{-1} = \frac{1}{-Z_0 - Z_1}\begin{bmatrix} 1 & -Z_0 \\ -1 & -Z_0 \end{bmatrix} = \frac{1}{2Z_0}\begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix}. \tag{10}$$

Note this is LeVeque Eq. 2.66. The initial conditions of the characteristic variables w^1 and w^2 are given in terms of the physical initial conditions by

$$\left[r^1 | r^2\right] \begin{bmatrix} w^1(x) \\ w^2(x) \end{bmatrix} = \begin{bmatrix} p_0(x) \\ u_0(x) \end{bmatrix} \tag{11}$$

Which gives (when multiplied out)

$$\begin{bmatrix} w^1(x) \\ w^2(x) \end{bmatrix} = \frac{1}{2Z_0}\begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix}\begin{bmatrix} p_0(x) \\ u_0(x) \end{bmatrix} = \frac{1}{2Z_0}\begin{bmatrix} -p_0(x) + Z_0 u_0(x) \\ p_0(x) + Z_0 u_0(x) \end{bmatrix} \tag{12}$$

Giving individually that

$$w^1(x) = \frac{1}{2Z_0}(-p_0(x) + Z_0 u_0(x)) \tag{13}$$

$$w^2(x) = \frac{1}{2Z_0}(p_0(x) + Z_0 u_0(x)) \tag{14}$$

Which is LeVeque Eq 2.67. Then with these initial conditions for our characteristic variables w^1 and w^2 we have our entire solution given by

$$q(x,t) = w^1(x + c_0 t) r^1 + w^2(x - c_0 t) r^2 \tag{15}$$

or in terms of w^1 and w^2

$$q(x,t) = \frac{1}{2Z_0}(-p_0(x + c_0 t) + Z_0 u_0(x + c_0 t))\begin{pmatrix} -Z_0 \\ 1 \end{pmatrix} \tag{16}$$

$$+ \frac{1}{2Z_0}(p_0(x - c_0 t) + Z_0 u_0(x - c_0 t))\begin{pmatrix} Z_0 \\ 1 \end{pmatrix} \tag{17}$$

Which gives for $q(x,t)$

$$q(x,t) = \frac{1}{2Z_0}\begin{pmatrix} Z_0 p_0(x + c_0 t) - Z_0^2 u_0(x + c_0 t) + Z_0 p_0(x - c_0 t) + Z_0^2 u_0(x - c_0 t) \\ -p_0(x + c_0 t) + Z_0 u_0(x + c_0 t) + p_0(x - c_0 t) + Z_0 u_0(x - c_0 t) \end{pmatrix} \tag{18}$$

or

$$q(x,t) = \frac{1}{2Z_0}\left(\begin{array}{c} Z_0(p(x+c_0t)+p(x-c_0t)) - Z_0^2(u(x+c_0t)-u(x-c_0t)) \\ -(p(x+c_0t)-p(x-c_0t)) + Z_0(u(x+c_0t)+u(x-c_0t)) \end{array}\right). \quad (19)$$

Finally we can extract components of the solution $q(x,t)$ obtaining

$$\begin{aligned} p(x,t) &= \frac{1}{2}(p(x+c_0t)+p(x-c_0t)) - \frac{Z_0}{2}(u(x+c_0t)+u(x-c_0t)) \\ u(x,t) &= -\frac{1}{2Z_0}(p(x+c_0t)-p(x-c_0t)) + \frac{1}{2}(u(x+c_0t)+u(x-c_0t)) \end{aligned}$$

Which is LeVeque Eq. 2.68.

Problem Solutions:

Chapter 2 (Conservation Laws and Differential Equations)

Problem 2.1 (Linear acoustics in terms of u and p)

LeVeque Eq 2.47 is given by

$$\begin{aligned} \tilde{\rho}_t + (\widetilde{\rho u})_x &= 0 \\ (\widetilde{\rho u})_t + (-u_0^2 + P'(\rho_0))\tilde{\rho}_x + 2u_0(\widetilde{\rho u})_x &= 0\,. \end{aligned} \quad (20)$$

To convert this set of equations into ones involving the variables u and p remember that linearizing about the constant state (ρ_0, p_0) gives a pressure perturbation $\tilde{p}$ that is linearly related to the density perturbation $\tilde{\rho}$ via, $\tilde{p} \approx P'(\rho_0)\tilde{\rho}$. Since we want to eliminate $\tilde{\rho}$ in favor of $\tilde{p}$ we solve for $\tilde{\rho}$ to obtain

$$\tilde{\rho} \approx \frac{\tilde{p}}{P'(\rho_0)}\,.$$

The linearization of $\widetilde{\rho u}$ in terms of $\tilde{u}$ and $\tilde{\rho}$ gives

$$\widetilde{\rho u} = u_0\tilde{\rho} + \rho_0\tilde{u}\,,$$

so that the conservation of mass equation in LeVeque Eq. 2.47 can now be written with these substitutions as

$$\frac{1}{P'(\rho_0)}\tilde{p}_t + u_0\tilde{\rho}_x + \rho_0\tilde{u}_x = 0\,.$$

Another application of the relationship $\tilde{\rho} \approx \frac{\tilde{p}}{P'(\rho_0)}$ replaces the spatial derivative $\tilde{\rho}_x$ in the above to give

$$\frac{1}{P'(\rho_0)}\tilde{p}_t + \frac{u_0}{P'(\rho_0)}\tilde{p}_x + \rho_0 \tilde{u}_x = 0\,.$$

Which after multiplying the equation by $P'(\rho_0)$ gives

$$\tilde{p}_t + u_0 \tilde{p}_x + \rho_0 P'(\rho_0)\tilde{u}_x = 0\,. \tag{21}$$

In a similar way the conservation of momentum equation can be written as

$$u_0 \tilde{\rho}_t + \rho_0 \tilde{u}_t + (-u_0^2 + P'(\rho_0))\frac{\tilde{p}_x}{P'(\rho_0)} + 2u_0\left(u_0\tilde{\rho}_x + \rho_0\tilde{u}_x\right) = 0\,.$$

Performing again the same substitution (ρ for p) in the above gives

$$\frac{u_0}{P'(\rho_0)}\tilde{p}_t + \rho_0 \tilde{u}_t + (-u_0^2 + P'(\rho_0))\frac{\tilde{p}_x}{P'(\rho_0)} + 2u_0\left(\frac{u_0\tilde{p}_x}{P'(\rho_0)} + \rho_0\tilde{u}_x\right) = 0\,.$$

Finally, replacing the time derivatives of $\tilde{p}$ with spatial derivatives from the conservation of mass equation 21 we have the left hand side of the above equation equal to

$$\begin{gathered}\frac{u_0}{P'(\rho_0)}(-u_0\tilde{p}_x - \rho_0 P'(\rho_0)\tilde{u}_x) + \\ \rho_0\tilde{u}_t + (-u_0^2 + P'(\rho_0))\frac{\tilde{p}_x}{P'(\rho_0)} + \\ 2u_0\left(u_0\frac{\tilde{p}_x}{P'(\rho_0)} + \rho_0\tilde{u}_x\right)\,.\end{gathered}$$

This simplifies to

$$\rho_0\tilde{u}_t + u_0\rho_0\tilde{u}_x + \tilde{p}_x = 0\,,$$

or dividing by ρ_0 we have

$$\tilde{u}_t + u_0\tilde{u}_x + \frac{1}{\rho_0}\tilde{p}_x = 0\,. \tag{22}$$

Now defining the bulk modulus of compressibility (as in the book) as $K_0 = \rho_0 P'(\rho_0)$ the equations 21 and 22 above become

$$\begin{aligned}\tilde{p}_t + u_0\tilde{p}_x + K_0\tilde{u}_x &= 0 \\ \tilde{u}_t + \frac{1}{\rho_0}\tilde{p}_x + u_0\tilde{u}_x &= 0\,.\end{aligned} \tag{23}$$

This is the desired equation. In a matrix form this can be written as

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u_0 & K_0 \\ \frac{1}{\rho_0} & u_0 \end{pmatrix} \begin{pmatrix} P \\ u \end{pmatrix} = 0\,,$$

which is LeVeque Eq. 2.50.

Problem 2.2 (smooth manipulations of the shallow water equations)

LeVeque Eq. 2.38 is

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= 0 \end{aligned} \tag{24}$$

Part (a): Given a functional relationship between p and ρ of the form $p = P(\rho)$, the derivative of p with respect to ξ (an arbitrary variable that could be either x or t depending on needed context) is given by

$$\frac{\partial p}{\partial \xi} = P'(\rho) \frac{\partial \rho}{\partial \xi}\,,$$

so the ρ derivative with respect to ξ is then given by

$$\frac{\partial \rho}{\partial \xi} = \frac{1}{P'(\rho)} \frac{\partial p}{\partial \xi}\,. \tag{25}$$

Now the first equation in 24 becomes using the product rule to expand $(\rho u)_x$

$$\rho_t + \rho_x u + \rho u_x = 0\,.$$

When we replace the time and spatial derivatives of ρ with corresponding ones for p using equation 25 we obtain

$$p_t + u p_x + \rho P'(\rho) u_x = 0\,. \tag{26}$$

In the second equation in 24 using the product to expand the derivatives gives

$$\rho_t u + u_t \rho + \rho_x u^2 + 2 u u_x \rho + P'(\rho) \rho_x = 0\,.$$

Next replacing the time and spatial derivatives of ρ with corresponding ones for p using equation 25 gives

$$\frac{p_t}{P'(\rho)} u + u_t \rho + \frac{p_x}{P'(\rho)} u^2 + 2 u u_x \rho + p_x = 0$$

finally by multiplying both sides of the equation above by $P'(\rho)$ we arrive at

$$up_t + \rho P'(\rho)u_t + u^2 p_x + 2\rho u P'(\rho)u_x + P'(\rho)p_x = 0\,.$$

We can simplify the above further by inserting the expression for p_t found in the conservation of mass equation 26. By doing so we have

$$u(-up_x - \rho P'(\rho)u_x) + \rho P'(\rho)u_t + u^2 p_x + 2\rho u P'(\rho)u_x + P'(\rho)p_x = 0$$

which simplifies to

$$\rho P'(\rho)u_t + \rho u P'(\rho)u_x + P'(\rho)p_x = 0\,,$$

or after dividing by $P'(\rho)$ we have

$$u_t + uu_x + \frac{1}{\rho}p_x = 0\,. \tag{27}$$

Note that equations 26 and 27 are the system we were requested to find. To demonstrate that a linearization of these equations about $\rho_0, u_0, p_0 = P(\rho_0)$ gives LeVeque Eq. 2.47 we define all our unknowns in terms of a base state and a perturbation to that state as follows

$$u = u_0 + \tilde{u} \tag{28}$$

$$\rho = \rho_0 + \tilde{\rho}\,. \tag{29}$$

With these definitions, the pressure can be evaluated in terms of a base state and an offset by using a Taylor expansion. Specifically we have

$$p = P(\rho_0 + \tilde{\rho}) = P(\rho_0) + P'(\rho_0)\tilde{\rho} + O(\tilde{\rho}^2) = p_0 + \tilde{p} \tag{30}$$

Where the last equation can be thought of as *defining* the offset of p in terms of the offset of ρ as $\tilde{p} \equiv P'(\rho_0)\tilde{\rho}$. With these substitutions for p, u, and ρ the quasi-linear conservation of mass equation 26 becomes

$$\tilde{p}_t + (u_0 + \tilde{u})\tilde{p}_x + (\rho_0 + \tilde{\rho})P'(\rho_0 + \tilde{\rho})\tilde{u}_x = 0\,.$$

Taylor expanding the $P'(\rho_0 + \tilde{\rho})$ term to second order we obtain

$$\tilde{p}_t + (u_0 + \tilde{u})\tilde{p}_x + (\rho_0 + \tilde{\rho})\left(P'(\rho_0) + P''(\rho_0)\tilde{\rho} + O(\tilde{\rho}^2)\right)\tilde{u}_x = 0\,.$$

Expanding products and keeping only first order terms (with respect to $\tilde{p}$, $\tilde{u}$, and $\tilde{\rho}$) we finally obtain

$$\tilde{p}_t + u_0 \tilde{p}_x + \rho_0 P'(\rho_0) \tilde{u}_x = 0 . \tag{31}$$

Now the quasi-linear conservation of momentum equation 27 with the substitutions from 28, 29, and 30 becomes

$$\tilde{u}_t + (u_0 + \tilde{u}) \tilde{u}_x + \frac{\tilde{p}_x}{\rho_0 + \tilde{\rho}} = 0 .$$

With this equation it is easier to derive asymptotics to all orders, first factor out ρ_0 from the denominator of the fraction in the last term above as follows

$$\tilde{u}_t + (u_0 + \tilde{u}) \tilde{u}_x + \frac{1}{\rho_0} \frac{1}{\left(1 + \frac{\tilde{\rho}}{\rho_0}\right)} \tilde{p}_x = 0 ,$$

then expand the fraction above in terms of the Taylor series for $\frac{1}{1+x}$ for small x, giving

$$\tilde{u}_t + (u_0 + \tilde{u}) \tilde{u}_x + \frac{1}{\rho_0} \sum_{k=0}^{\infty} \left(\frac{\tilde{\rho}}{\rho_0} \right)^k \tilde{p}_x = 0 .$$

This gives an expression valid to all orders of $\tilde{\rho}$ as long as $\frac{\tilde{\rho}}{\rho_0}$ is small. With this expression, to obtain the requested equation valid up to first order we keep only the $k = 0$ term above (and drop the second order term $\tilde{u}\tilde{u}_x$) to obtain

$$\tilde{u}_t + u_0 \tilde{u}_x + \frac{1}{\rho_0} \tilde{p}_x = 0 \tag{32}$$

Thus our linearized system (combining equations 31 and 32) becomes (dropping the tildes)

$$p_t + u_0 p_x + \rho_0 P'(\rho_0) u_x = 0 \tag{33}$$

$$u_t + u_0 u_x + \frac{1}{\rho_0} p_x = 0 . \tag{34}$$

Finally in the notation of a linear system we have

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} u_0 & \rho_0 P'(\rho_0) \\ \frac{1}{\rho_0} & u_0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0 .$$

Defining the bulk modulus of compressibility K_0 as $K_0 \equiv \rho_0 P'(\rho_0)$ this expression is equivalent to LeVeque Eq. 2.50 as was asked to be shown.
Part (b): LeVeque Eq. 2.122 is

$$\begin{aligned} p_t + u p_x + \rho P'(\rho) u_x &= 0 \\ u_t + \frac{1}{\rho} p_x + u u_x &= 0 \,. \end{aligned} \tag{35}$$

Which in matrix quasi-linear form is given by

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} u & \rho P'(\rho) \\ \frac{1}{\rho} & u \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0 \,. \tag{36}$$

The characteristic speeds of this system are given by the eigenvalues of the coefficient matrix or the solutions λ of the following characteristic equation

$$\begin{vmatrix} u - \lambda & \rho P'(\rho) \\ \frac{1}{\rho} & u - \lambda \end{vmatrix} = 0 \,.$$

On expanding the determinant we have

$$(u - \lambda)^2 - P'(\rho) = 0 \,,$$

and finally solving for λ we obtain

$$\lambda^{1,2} = u \mp \sqrt{P'(\rho)} \,. \tag{37}$$

For hyperbolicity, each eigenvalue must be real. From equation 37 above this requires $P'(\rho) > 0$, as was asked to be shown.

A similar derivation of the characteristic speed in terms of the *conservative* variables, given by LeVeque Eq. 2.38 and LeVeque Eq. 2.40 will now show that the same characteristic speeds are present no matter what formulation of the unknown variables (conservative or primitive) we use to represent these equations. To verify this claim, we first recall the conservative equations LeVeque Eq. 2.38 which are

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= 0 \,. \end{aligned} \tag{38}$$

Defining a state vector q of conservative unknowns as

$$q = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \end{bmatrix} \,, \tag{39}$$

our conservative representation of the equations above define a flux function $f(q)$ as

$$f(q) = \begin{bmatrix} q^2 \\ \frac{(q^2)^2}{q^1} + P(q^1) \end{bmatrix} . \tag{40}$$

These two expression combine to gives a quasi-linear representation $q_t + f'(q)q_x = 0$ in conservative variables which is given by

$$\begin{bmatrix} q^1 \\ q^2 \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ -\frac{(q^2)^2}{(q^1)^2} + P'(q^1) & \frac{2q^2}{q^1} \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}_x = 0 . \tag{41}$$

The characteristic speeds of this quasi-linear system is given by the solutions λ of

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{(q^2)^2}{(q^1)^2} + P'(q^1) & \frac{2q^2}{q^1} - \lambda \end{vmatrix} = 0 ,$$

or upon expanding the determinant we have

$$-\lambda(\frac{2q^2}{q^1} - \lambda) + \left(\frac{q^2}{q^1}\right)^2 - P'(q^1) = 0 .$$

This yields the following quadratic expression for λ

$$\lambda^2 - 2\lambda\left(\frac{q^2}{q^1}\right) + \left(\frac{q^2}{q^1}\right)^2 - P'(q^1) = 0 .$$

Solving this quadratic equation (using the formula form high school) we obtain

$$\lambda = \frac{2\left(\frac{q^2}{q^1}\right) \pm \sqrt{4\left(\frac{q^2}{q^1}\right)^2 - 4\left(\left(\frac{q^2}{q^1}\right)^2 - P'(q^1)\right)}}{2} .$$

By canceling terms in the square root we find

$$\lambda = \frac{2\left(\frac{q^2}{q^1}\right) \pm \sqrt{4P'(\rho)}}{2} = \frac{q^2}{q^1} \pm \sqrt{P'(\rho)} = u \pm \sqrt{P'(\rho)} . \tag{42}$$

This is the same as the expression derived earlier, showing the equivalence of the eigenvalues with respect to a conservative v.s. nonconservative formulation as claimed.

Problem 2.3 (the characteristic speeds of the second order wave equation)

LeVeque Eq. 2.77 is

$$\tilde{A} = \begin{bmatrix} 0 & c_0^2 \\ 1 & 0 \end{bmatrix} .$$

The eigenvalues of $\tilde{A}$ are given by solving for λ in $\tilde{A}$'s characteristic equation given by $|\tilde{A} - \lambda I| = 0$. For our matrix $\tilde{A}$ given above this characteristic equation is given by

$$\begin{vmatrix} -\lambda & c_0^2 \\ 1 & -\lambda \end{vmatrix} = 0 .$$

On expanding the determinant we obtain $\lambda^2 - c_0^2 = 0$, which has two solutions given by $\pm c_0$. Denoting by λ^1 the smaller eigenvalue and by λ^2 the larger we have $\lambda^1 = -c_0$ and $\lambda^2 = +c_0$.

The eigenvectors of $\tilde{A}$ are given by finding a vector v that satisfies the the system $\tilde{A}v = \lambda v$. For $\lambda^1 = -c_0$ this linear system becomes (after writing it as $(\tilde{A} - I)v = 0$)

$$\left(\begin{bmatrix} 0 & c_0^2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -c_0 & 0 \\ 0 & -c_0 \end{bmatrix} \right) \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = 0 ,$$

or performing the subtraction we obtain

$$\begin{bmatrix} -c_0 & c_0^2 \\ 1 & -c_0 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = 0 .$$

Now dividing the first equation by $-c_0$ gives

$$\begin{bmatrix} 1 & -c_0 \\ 1 & -c_0 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = 0 ,$$

which gives two equations enforcing the repeated constraint that $v_1^1 - c_0 v_2^1 = 0$ or $v_1^1 = c_0 v_2^1$ and shows that the eigenvector corresponding to eigenvalue $-c_0$ is given by

$$v^1 = \begin{bmatrix} -c_0 \\ 1 \end{bmatrix} . \tag{43}$$

By analogy we have that v^2 is given by

$$v^2 = \begin{bmatrix} +c_0 \\ 1 \end{bmatrix} . \tag{44}$$

We now turn to finding the similarity transformation that relates this matrix $\tilde{A}$ to A defined by LeVeque Eq. 2.51 where A is given by

$$A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}.$$

Here $K_0 \equiv \rho_0 P'(\rho_0) = \rho_0 c_0^2$ since $c_0^2 = P'(\rho)$ for the linear acoustic equations. With this definition of K_0 an equivalent formulation of A is given by

$$A = \begin{bmatrix} 0 & \rho_0 c_0^2 \\ 1/\rho_0 & 0 \end{bmatrix}.$$

From the discussion in Section 2.8 the eigenvalues of A (when $u_0 = 0$) are the same as those of $\tilde{A}$. Performing the characteristic decomposition of each we would then have that

$$\begin{aligned} A &= R\Lambda R^{-1} \\ \tilde{A} &= \tilde{R}\Lambda\tilde{R}^{-1} \end{aligned}$$

where the matrix Λ is the common two by two diagonal matrix with diagonal elements given by $-c_0$ and c_0. The eigenvector matrices R and $\tilde{R}$ will in general be different. From this characteristic decomposition solving for Λ using the second equation gives $\Lambda = \tilde{R}^{-1}\tilde{A}\tilde{R}$, which when put in the first equation gives

$$A = (R\tilde{R}^{-1})\tilde{A}(\tilde{R}R^{-1}).$$

If we define a matrix S by $S = \tilde{R}R^{-1}$, we see that A and $\tilde{A}$ are related by $A = S^{-1}\tilde{A}S$. From these simple manipulations one can obtain a similarity transformation for two similar matrices by diagonalizing each and combining the matrices of eigenvectors in the appropriate way.

Using the eigenvectors of A given in Section 2.8 we have that *one* similarity transformation is given by

$$\begin{aligned} S &= \tilde{R}R^{-1} \\ &= \begin{bmatrix} -c_0 & c_0 \\ 1 & 1 \end{bmatrix} \left(\frac{1}{2\rho_0 c_0}\right) \begin{bmatrix} -1 & \rho_0 c_0 \\ 1 & \rho_0 c_0 \end{bmatrix} \\ &= \begin{bmatrix} 1/\rho_0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Another method (although much more tedious) of obtaining a similarity transformation is to remember that if $\tilde{A}$ and A are related by a similarity transformation then this means that there exists an invertible matrix S such that

$$S^{-1}\tilde{A}S = A$$

or equivalently, by multiplying by S on both sides of this equation we obtain

$$\tilde{A}S = SA\,. \tag{45}$$

To find a candidate matrix S define it in terms of four unknowns a, b, c, and d as

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then equation 45 becomes in terms of a, b, c, and d the following

$$\begin{bmatrix} 0 & c_0^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & \rho_0 c_0^2 \\ 1/\rho_0 & 0 \end{bmatrix}.$$

By multiplying everything out we will attempt to form a set of four linear equations and four unknowns for the variables a, b, c, and d and then solve for them using standard techniques. We begin by multiplying the matrices together on both sides of the above to obtain

$$\begin{bmatrix} cc_0^2 & dc_0^2 \\ a & b \end{bmatrix} = \begin{bmatrix} b/\rho_0 & a\rho_0 c_0^2 \\ d/\rho_0 & c\rho_0 c_0^2 \end{bmatrix}.$$

Equating elements in each matrix we obtain four equations given by

$$cc_0^2 = b/\rho_0 \tag{46}$$
$$dc_0^2 = a\rho_0 c_0^2 \tag{47}$$
$$a = d/\rho_0 \tag{48}$$
$$b = c\rho_0 c_0^2\,, \tag{49}$$

which when written as a homogeneous system (a system set equal to zero) gives

$$\begin{aligned} b/\rho_0 - cc_0^2 &= 0 \\ a\rho_0 c_0^2 - dc_0^2 &= 0 \\ a - d/\rho_0 &= 0 \\ b - c\rho_0 c_0^2 &= 0\,. \end{aligned}$$

The benefit of this is that all unknowns can be written in matrix notation as

$$C \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \equiv \begin{bmatrix} 0 & 1/\rho_0 & -c_0^2 & 0 \\ \rho_0 c_0^2 & 0 & 0 & -c_0^2 \\ 1 & 0 & 0 & -1/\rho_0 \\ 0 & 1 & -\rho_0 c_0^2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0\,.$$

Where we have defined the coefficient matrix C in the above expression. To determine if this system uniquely determines a,b,c, and d we will evaluate the determinant of C to determine if the matrix itself is invertible. The determinant of this coefficient matrix C is given by expanding in terms of minors along the first column

$$|C| = -\rho_0 c_0^2 \begin{vmatrix} 1/\rho_0 & -c_0^2 & 0 \\ 0 & 0 & -1/\rho_0 \\ 1 & -\rho_0 c_0^2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1/\rho_0 & -c_0^2 & 0 \\ 0 & 0 & -c_0^2 \\ 1 & -\rho_0 c_0^2 & 0 \end{vmatrix}\,,$$

or again expanding in terms of minors

$$|C| = -\rho_0 c_0^2 \frac{1}{\rho_0} \begin{vmatrix} 1/\rho_0 & -c_0^2 \\ 1 & -\rho_0 c_0^2 \end{vmatrix} + c_0^2 \begin{vmatrix} 1/\rho_0 & -c_0^2 \\ 1 & -\rho_0 c_0^2 \end{vmatrix}\,,$$

Which finally gives for $|C|$ the following

$$|C| = -c_0^2(-c_0^2 + c_0^2) + c_0^2(-c_0^2 + c_0^2) = 0\,.$$

Thus the system as given is of smaller dimension then originally thought ($4x4$) and we can reduce the number of unknowns (and find a particular solution) by taking some of the coefficients to be known. We will choose a value that make the algebra easier for example let $a = 1/\rho_0$ then from equation 48 above we have $d = 1$. With this substitution only equations 46 and 49 remain

$$\begin{aligned} c c_0^2 &= b/\rho_0 \\ b &= c\rho_0 c_0^2 \end{aligned}$$

Which again results in an underdetermined system as can be seen by manipulations similar to what was performed above. Specifying c, we $c = 1$ then obtain $b = \rho_0 c_0^2$, and finally putting a, b, c, and d into the expression for S we obtain

$$S = \begin{bmatrix} 1/\rho_0 & \rho_0 c_0^2 \\ 1 & 1 \end{bmatrix}\,.$$

This S has an inverse given by

$$S^{-1} = \frac{\rho_0}{1-(\rho_0 c_0)^2} \begin{bmatrix} 1 & -\rho_0 c_0^2 \\ -1 & 1/\rho_0 \end{bmatrix}.$$

As a check, we can indeed verify that $S^{-1}\tilde{A}S = A$. The product and the algebra follow

$$\begin{aligned}
S^{-1}\tilde{A}S &= \frac{\rho_0}{1-(\rho_0 c_0)^2} \begin{bmatrix} 1 & -\rho_0 c_0^2 \\ -1 & 1/\rho_0 \end{bmatrix} \begin{bmatrix} 0 & c_0^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\rho_0 & \rho_0 c_0^2 \\ 1 & 1 \end{bmatrix} \\
&= \frac{\rho_0}{1-(\rho_0 c_0)^2} \begin{bmatrix} 1 & -\rho_0 c_0^2 \\ -1 & 1/\rho_0 \end{bmatrix} \begin{bmatrix} c_0^2 & c_0^2 \\ 1/\rho_0 & \rho_0 c_0^2 \end{bmatrix} \\
&= \frac{\rho_0}{1-(\rho_0 c_0)^2} \begin{bmatrix} c_0^2 - c_0^2 & c_0^2 - \rho_0^2 c_0^4 \\ -c_0^2 + 1/\rho_0^2 & -c_0^2 + c_0^2 \end{bmatrix} \\
&= \frac{\rho_0}{1-(\rho_0 c_0)^2} \begin{bmatrix} 0 & c_0^2(1-\rho_0^2 c_0^2) \\ 1/\rho_0^2(1-c_0^2\rho_0^2) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \rho_0 c_0^2 \\ 1/\rho_0 & 0 \end{bmatrix}.
\end{aligned}$$

Showing the requested similarity between A and $\tilde{A}$. Since a similarity transformation is not unique (multiplication by a non-zero constant produces another one) the fact that we found two different similarity matrices is not incorrect. In fact, in setting up the four by four linear system had we chosen our constants a, b, c, and d differently, i.e. $a = 1/\rho_0$, $b = 0$, $c = 0$, $d = 1$, we would have obtained exactly the same similarity matrix as before.

Problem 2.4 (the eigensystem for linear acoustics in primitive variables)

LeVeque Eq. 2.46 is

$$A = \begin{bmatrix} 0 & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 \end{bmatrix},$$

which has eigenvalues λ given by the solution to the characteristic equation for A or

$$\begin{vmatrix} -\lambda & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 - \lambda \end{vmatrix} = 0\,.$$

On expanding the determinant above we obtain

$$-\lambda(2u_0 - \lambda) + (u_0^2 - P'(\rho_0)) = 0\,,$$

which simplifies to give the following quadratic equation

$$\lambda^2 - 2u_0\lambda + u_0^2 - P'(\rho_0) = 0\,.$$

Solving this using the quadratic formula we obtain

$$\lambda = \frac{2u_0 \pm \sqrt{4u_0^2 - 4(u_0^2 - P'(\rho_0))}}{2} = \frac{2u_0 \pm \sqrt{4P'(\rho_0)}}{2} = u_0 \pm \sqrt{P'(\rho_0)}\,.$$

Ordering the eigenvalues such that $\lambda^1 < \lambda^2$ we have that,

$$\begin{aligned} \lambda^1 &= u_0 - \sqrt{P'(\rho_0)} \\ \lambda^2 &= u_0 + \sqrt{P'(\rho_0)} \end{aligned}$$

Defining $c_0 \equiv \sqrt{P'(\rho_0)}$, both eigenvalues above agree with the expression given by LeVeque Eq. 2.57, as we were requested to show.

To compute the eigenvectors, the first eigenvector v^1 is given by finding a v in the nullspace of the operator $(A - \lambda^1 I)$, which in matrix form is

$$\left[\begin{array}{cc} -u_0 + \sqrt{P'(\rho_0)} & 1 \\ -u_0^2 + P'(\rho_0) & u_0 + \sqrt{P'(\rho_0)} \end{array} \right] \left[\begin{array}{c} v_1^1 \\ v_2^1 \end{array} \right] = 0\,,$$

or by factoring the element in the $(2,1)$ position we have

$$\left[\begin{array}{cc} -u_0 + \sqrt{P'(\rho_0)} & 1 \\ \left(-u_0 + \sqrt{P'(\rho)}\right)\left(+u_0 + \sqrt{P'(\rho)}\right) & u_0 + \sqrt{P'(\rho_0)} \end{array} \right] \left[\begin{array}{c} v_1^1 \\ v_2^1 \end{array} \right] = 0\,.$$

Now since the second row is a multiple of the first row we have a single constraint on the components of the vector v^1 of

$$(-u_0 + \sqrt{P'(\rho_0)})v_1^1 + v_2^1 = 0\,.$$

In vector form our first eigenvector is given by

$$v^1 = \left[\begin{array}{c} 1 \\ u_0 - \sqrt{P'(\rho_0)} \end{array} \right]\,. \tag{50}$$

By analogy, we have for the second eigenvector v^2 the expression

$$v^2 = \begin{bmatrix} 1 \\ u_0 + \sqrt{P'(\rho_0)} \end{bmatrix} . \tag{51}$$

We will now compute the similarity matrix S between LeVeque Eq. 2.46 (denoted by A) and LeVeque Eq. 2.50 (denoted by $\tilde{A}$). Writing $\sqrt{P'(\rho_0)} = c_0$ and $K_0 = \rho_0 c_0^2$ our two matrices in terms of ρ_0 and c_0 become the following

$$A = \begin{bmatrix} 0 & 1 \\ -u_0^2 + c_0^2 & 2u_0 \end{bmatrix} \quad \text{and} \quad \tilde{A} = \begin{bmatrix} u_0 & \rho_0 c_0^2 \\ 1/\rho_0 & u_0 \end{bmatrix} .$$

To compute the similarity matrix S, we recall from Problem 2.3 once we have diagonalized both A and $\tilde{A}$ that a similiarity matrix between A and $\tilde{A}$ is given by

$$S = \tilde{R} R^{-1} .$$

Here the similarity transformation implied is $A = S^{-1} \tilde{A} S$. We can therefore compute this similarity transformation as

$$\begin{aligned} S &= \tilde{R} R^{-1} \\ &= \begin{bmatrix} -\rho_0 c_0 & \rho_0 c_0 \\ 1 & 1 \end{bmatrix} \left(\frac{1}{2c_0} \right) \begin{bmatrix} u_0 + c_0 & -1 \\ -u_0 + c_0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\rho_0 u_0 & \rho_0 \\ 1 & 0 \end{bmatrix} . \end{aligned}$$

Here we have used the right eigenvectors from $\tilde{A}$ which are computed in Section 2.8.

As a check, we can indeed verify that $S^{-1} \tilde{A} S = A$. The product and the algebra follow

$$\begin{aligned} S^{-1} \tilde{A} S &= \frac{1}{\rho_0} \begin{bmatrix} 0 & \rho_0 \\ 1 & \rho_0 u_0 \end{bmatrix} \begin{bmatrix} u_0 & \rho_0 c_0^2 \\ 1/\rho_0 & u_0 \end{bmatrix} \begin{bmatrix} -\rho_0 u_0 & \rho_0 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{\rho_0} \begin{bmatrix} 1 & \rho_0 u_0 \\ 2u_0 & \rho_0 (c_0^2 + u_0^2) \end{bmatrix} \begin{bmatrix} -\rho_0 u_0 & \rho_0 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{\rho_0} \begin{bmatrix} \rho_0 u_0 - \rho_0 u_0 & \rho_0 \\ -2\rho_0 u_0^2 \rho_0 (c_0^2 + u_0^2) & 2\rho_0 u_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ c_0^2 - u_0^2 & 2u_0 \end{bmatrix} . \end{aligned}$$

Showing the requested similarity between A and $\tilde{A}$.

Problem 2.5 (constraints for hyperbolicity for 1d elastic waves)

LeVeque Eq. 2.91 is given by

$$\begin{aligned} \epsilon_t^{11} - u_x &= 0 \\ \rho u_t - \sigma_x^{11} &= 0\,. \end{aligned} \tag{52}$$

Dropping the superscript of "11" for notational simplicity and assuming that $\sigma = \sigma(\epsilon)$, for smooth solutions the above can be written as

$$\begin{aligned} \epsilon_t - u_x &= 0 \\ u_t - \frac{1}{\rho}\sigma'(\epsilon)\epsilon_x &= 0\,, \end{aligned} \tag{53}$$

or in matrix form by

$$\begin{bmatrix} \epsilon \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ -\frac{\sigma'(\epsilon)}{\rho} & 0 \end{bmatrix} \begin{bmatrix} \epsilon \\ u \end{bmatrix}_x = 0\,.$$

To be hyperbolic it is necessary and sufficient that the Jacobian matrix defined by

$$A = \begin{bmatrix} 0 & -1 \\ -\frac{\sigma'(\epsilon)}{\rho} & 0 \end{bmatrix}\,,$$

have real eigenvalues and complete set of eigenvectors. We now compute the eigenvalues. The eigenvalues are the solutions λ to the following equation $|A - \lambda I| = 0$, which for this system looks like

$$\begin{vmatrix} -\lambda & -1 \\ -\frac{\sigma'(\epsilon)}{\rho} & -\lambda \end{vmatrix} = 0\,.$$

By expanding the determinant the above becomes

$$\lambda^2 - \frac{\sigma'(\epsilon)}{\rho} = 0\,,$$

which has solutions given by

$$\lambda^{1,2} = \mp\sqrt{\frac{\sigma'(\epsilon)}{\rho}}\,.$$

Thus λ will be real and our system hyperbolic if and only if $\sigma'(\epsilon) > 0$. With the eigenvalues above the eigenvector for $\lambda^1 = -\sqrt{\frac{\sigma'(\epsilon)}{\rho}}$ are given by looking for the nullspace to the operator $A - \lambda^1 I$ which in this case is given by

$$\begin{bmatrix} \sqrt{\frac{\sigma'}{\rho}} & -1 \\ -\frac{\sigma'}{\rho} & \sqrt{\frac{\sigma'}{\rho}} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = 0\,.$$

This system gives the constraint on the vector v that

$$\sqrt{\frac{\sigma'}{\rho}} v^1 - v^2 = 0\,,$$

or

$$v^2 = \sqrt{\frac{\sigma'}{\rho}} v^1\,.$$

This gives as the eigenvector for λ^1

$$v = \begin{bmatrix} 1 \\ \sqrt{\frac{\sigma'}{\rho}} \end{bmatrix}$$

In the same way, we have for λ^2 the following eigenvector

$$v = \begin{bmatrix} 1 \\ -\sqrt{\frac{\sigma'}{\rho}} \end{bmatrix}\,.$$

Problem 2.6 (consistency relationships in Lagrangian coordinates)

The easiest way to see this is to consider LeVeque Eq. 2.103 which is

$$\int_{\xi_1}^{\xi_2} V(\xi, t) d\xi = X(\xi_2, t) - X(\xi_1, t)\,,$$

and to differentiate with respect to ξ_2. When this is done one obtains

$$V(\xi_2, t) = X_{\xi_2}(\xi_2, t)\,,$$

or the requested identity.

Problem 2.7 (constraints for hyperbolicity for the p-system)

LeVeque Eq. 2.108 is given by

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= 0\,. \end{aligned} \tag{54}$$

This has a quasilinear form given by

$$\left[\begin{array}{c} v \\ u \end{array}\right]_t + \left[\begin{array}{cc} 0 & -1 \\ p'(v) & 0 \end{array}\right] \left[\begin{array}{c} v \\ u \end{array}\right]_x = 0\,.$$

The eigenvalues of the Jacobian coefficient matrix are given by the solutions λ to the characteristic equation given by

$$\left|\begin{array}{cc} -\lambda & -1 \\ p'(v) & -\lambda \end{array}\right| = 0\,.$$

Upon expanding the determinant above we have

$$\lambda^2 + p'(v) = 0\,,$$

which gives for λ the following

$$\lambda^{1,2} = \mp\sqrt{-p'(v)}\,.$$

To be hyperbolic means that λ is real or that the function $p(\cdot)$ must satisfy $-p'(v) > 0$ equivalently $p'(v) < 0$ for all v.

Problem 2.8 (wave speeds for the 1d isothermal equations)

LeVeque Eq 2.38

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= 0\,, \end{aligned} \tag{55}$$

which when $P(\rho) = a^2\rho$, (a is constant) are called the isothermal equations. With a general $P(\rho)$ relationship they are called the shallow water equations.
Part (a): The linearized version of the shallow water equations have a coefficient matrix A given by LeVeque Eq. 2.50 of

$$A = \left[\begin{array}{cc} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{array}\right]\,.$$

In this expression, the bulk modulus of compressibility is given by

$$K_0 = \rho_0 P'(\rho_0) = \rho_0 a^2 \,.$$

This then gives for the matrix A the following

$$A = \begin{bmatrix} u_0 & \rho_0 a^2 \\ 1/\rho_0 & u_0 \end{bmatrix} .$$

The eigenvalues of this matrix A are given by the solutions to the characteristic equation for A or $|A - \lambda I| = (u_0 - \lambda)^2 - a^2 = 0$, which has roots given by gives $\lambda = u_0 \mp a$. These are the Eulerian waves speeds.

Part (b): LeVeque Eq. 2.107 is given by

$$\begin{aligned} v_t - u_\xi &= 0 \\ u_t + p(v)_\xi &= 0 \,. \end{aligned} \tag{56}$$

In the Lagrangian form of the isothermal equation we have $p(v) = \frac{a^2}{v}$, which gives a quasilinear form for general $p = p(v)$ of

$$\begin{bmatrix} v \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}_\xi = 0 \,,$$

which in the specific functional p of v relation given here becomes

$$\begin{bmatrix} v \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ \frac{-a^2}{v^2} & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}_\xi = 0 \,,$$

since $p'(v) = -\frac{a^2}{v^2}$. Now the linearization of this quasilinear system can be obtained by evaluating the Jacobian above at the linearization point of (v_0, u_0) and gives

$$\begin{bmatrix} v \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ \frac{-a^2}{v_0^2} & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}_\xi = 0 \,,$$

From which the Jacobian above gives eigenvalues of $\lambda^{1,2} = \mp\sqrt{\frac{a^2}{v_0^2}} = \mp\frac{a}{v_0}$.

Chapter 3 (Characteristics and Riemann Problems for Linear Hyperbolic Equations)

Problem 3.1 (various linear Riemann problems)

This problem is focused on solving the linear hyperbolic equation $u_t + Au_x = 0$, for various A's.
Part (a): For the A of

$$A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix},$$

its eigenvalues are given by the solutions λ to $|A - \lambda I| = 0$ or

$$\begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = 0 \,.$$

Which gives $\lambda^{1,2} = \mp 2$. For the eigenvector for $\lambda^1 = -2$ we look for a vector r in the nullspace of the operator $A + 2I$ i.e. r^1 must satisfy the following

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} r^1 = 0 \,.$$

One such vector is

$$r^1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} . \tag{57}$$

For $\lambda^2 = +2$ our the matrix we look for the nullspace of is $A - 2I$ and we now look for a r^2 that satisfies

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} r^2 = 0 \,,$$

which gives for a vector in the nullspace (or the eigenvector of)

$$r^2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} . \tag{58}$$

The solution of the linear Riemann problem involves decomposing the jump in state $q_r - q_l$ in terms of eigenvectors, and is facilitated by using a matrix R (the a matrix of right eigenvectors) given by

$$R = [r^1 | r^2] = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix},$$

which has an inverse given by

$$R^{-1} = \frac{1}{-2-2}\begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix}.$$

Now our jump in state for this problem is given by

$$q_r - q_l = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

and so decomposing this jump into a linear combination of eigenvector involves solving for the vector α in

$$R\alpha = q_r - q_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The solution for α is simple and given by

$$\alpha = R^{-1}(q_r - q_l) = \frac{1}{4}\begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The solution $q(x,t)$ is then in terms of the α_i's given by

$$q(x,t) = q_l + \sum_{p:\lambda^p < x/t} \alpha^p r^p .$$

Since in this problem we only have two waves we will only have a state different than the left and right ones when x/t is between the two characteristic speeds. This *middle* state is given by

$$q_m = q_l + \alpha^1 r^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix}.$$

We can check this by computing the middle state from the right state in stead as

$$q_m = q_r - \alpha^2 r^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix},$$

giving the same thing.

Part (b): Here our A is given by

$$A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

which has eigenvalue and eigenvectors given by (these are computed in the same way as **Part (a)** above)

$$\begin{aligned}\lambda^1 &= -2 \quad \text{with} \quad r^1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ \lambda^2 &= +2 \quad \text{with} \quad r^2 = \begin{bmatrix} +2 \\ 1 \end{bmatrix} .\end{aligned}$$

Our coordinate transformation matrix R and its inverse, composed of the right eigenvalues of A, is then given by

$$R = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{with} \quad R^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} .$$

So to decompose our initial jump in state $q_r - q_l$ into components of the eigenvector basis α, we are looking for the solution to,

$$R\alpha = q_r - q_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} ,$$

which has α given by

$$\alpha = \frac{1}{4} \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

To compute the middle state we remember that to go from q_l to q_m we must go along a r^1 shock while to go from q_r to q_m must be along a r^2 shock. Each of these "paths" gives two possible ways of computing q_m. Using each we get the following two formulas for q_m

$$\begin{aligned}q_m &= q_l + \alpha^1 r^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 5/4 \end{bmatrix} \\ q_m &= q_r - \alpha^2 r^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} +2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 5/4 \end{bmatrix} .\end{aligned}$$

Part (c): For this part our coefficient matrix A is given by

$$A = \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix} .$$

Which has eigenvalues and vectors given by

$$\begin{aligned}
\lambda^1 &= -3 \quad \text{with} \quad r^1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\
\lambda^2 &= +3 \quad \text{with} \quad r^2 = \begin{bmatrix} +3 \\ 1 \end{bmatrix}.
\end{aligned}$$

With these eigenvectors the matrix of eigenvectors R and its inverse is given by

$$R = \begin{bmatrix} -3 & +3 \\ 1 & 1 \end{bmatrix} \quad \text{with} \quad R^{-1} = \frac{1}{6}\begin{bmatrix} -1 & +3 \\ 1 & 3 \end{bmatrix}.$$

Thus the initial Riemann problem jump in an eigenvector coordinate system α is given by $\alpha = R^{-1}(q_r - q_l)$, which in this case is

$$\alpha = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.$$

Given this decomposition, the state at any point in the x-t plane can be obtained by summing waves beginning from the left state q_l or the right state q_r from the following expression

$$q(x,t) = q_l + \sum_{p:\lambda^p < x/t} \alpha^p r^p = q_r - \sum_{p:\lambda^p \geq x/t} \alpha^p r^p$$

Now since $\lambda^1 < 0$ and $\lambda^2 > 0$, we can evaluate the first expression above at $x = 0$ to obtain a middle state of the following

$$q_m = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{p:\lambda^p < 0} \alpha^p r^p = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}.$$

We can check this numerical value by using the second expression (again evaluated at $x = 0$)

$$q_m = q_r + \sum_{p:\lambda^p \geq 0} \alpha^p r^p = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}.$$

To evaluate our solution at $t = 1$ we would consider $q(x,1) = q_l + \sum_{p:\lambda^p < x} \alpha^p r^p$ which gives

$$q(x,1) = \begin{cases} q_l & x < -3 \\ q_m & -3 < x < 3 \\ q_r & x > 3 \end{cases}$$

Part (d): For this part we have a coefficient matrix A given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Which has eigenvalues and eigenvectors given by

$$\lambda^1 = 0 \quad \text{and} \quad r^1 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$
$$\lambda^2 = 2 \quad \text{and} \quad r^2 = \begin{bmatrix} +1 \\ +1 \end{bmatrix}.$$

This means that the matrix of eigenvectors and its inverse is given by

$$R = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad R^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In this case, the initial jump in state has coefficients in terms of the eigenvectors of A as $\alpha = R^{-1}(q_r - q_l)$, which for this problem is explicitly given by

$$\alpha = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.$$

The solution of a linear hyperbolic problem at any point in the x-t plane is given by the usual expression of

$$q(x,t) = q_l + \sum_{p:\lambda^p < x/t} \alpha^p r^p = q^r - \sum_{p:\lambda^p \geq x/t} \alpha^p r^p.$$

Since $\lambda^1 = 0$ and $\lambda^2 = 2$, so to used the above to evaluate the middle state we should evaluate at the points $x/t = 1$. When used from the left state these values of x and t compute a middle state given by

$$q_m = q^l + \sum_{p:\lambda^p < 1} \alpha^p r^p = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}.$$

Checking our results using the other formula centered on q^r gives

$$q^m = q^r - \sum_{p:\lambda^p \geq 1} \alpha^r r^p = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix},$$

thus verifying our computations.
Part (e): In this case our matrix is given by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

which has only one distinct eigenvalue given by $\lambda = 2$. The two corresponding eigenvectors given by looking for the nullspace of the operator $A - 2I$, which is equivalent to solving for a nonzero vector r such that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} r = 0 \, .$$

This system has two linearly independent solutions given by

$$r^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad r^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

With these our matrix of eigenvectors R and its inverse is given by

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Computing our initial Riemann problem jump in state in terms of the eigenvectors of A requires solving $R\alpha = q_r - q_l$, or since $R = I$, we have

$$\alpha = q_r - q_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

The middle state solution to the Riemann problem can be written in two different ways as

$$q_m = q_l + \alpha^1 r^1 = q_r - \alpha^2 r^2 \, ,$$

which for this problem becomes

$$\begin{aligned} q_m &= q_l + \alpha^1 r^1 = q_l + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ q_m &= q_r - \alpha^2 r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \end{aligned}$$

We note that this state only occupies a sliver of points along the ray $x/t = 2$.

Part (f): The matrix A in this case is

$$A = \begin{bmatrix} 2 & 1 \\ 10^{-4} & 2 \end{bmatrix} = 0\,,$$

and has eigenvalues given by the solutions λ to $(2-\lambda)^2 = 10^{-4}$, which are $\lambda^{1,2} = 2 \mp 10^{-2}$. These two eigenvalues have eigenvectors given by

$$r^1 = \begin{bmatrix} -100 \\ 1 \end{bmatrix} \quad \text{and} \quad r^2 = \begin{bmatrix} +100 \\ 1 \end{bmatrix}\,,$$

so the matrix with columns the eigenvectors R (and its inverse) is given by

$$R = \begin{bmatrix} -100 & 100 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad R^{-1} = \frac{1}{200}\begin{bmatrix} -1 & 100 \\ 1 & 100 \end{bmatrix}$$

The initial jump in state has coefficients in terms of the eigenvectors of A given by $\alpha = R^{-1}(q_r - q_l)$, which for this problem is explicitly given by

$$\alpha = \frac{1}{200}\begin{bmatrix} -1 & 100 \\ 1 & 100 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{200}\begin{bmatrix} -101 \\ -99 \end{bmatrix}.$$

Our middle state is given by

$$q^m = q^l + \alpha^1 r^1\,,$$

which in this case gives

$$q^m = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{101}{200}\begin{bmatrix} -100 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{101}{2} \\ \frac{99}{200} \end{bmatrix}.$$

This results for q^m can be checked by computing it another way

$$q^m = q^r - \alpha^2 r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{99}{200}\begin{bmatrix} 100 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{101}{2} \\ \frac{99}{200} \end{bmatrix}.$$

Please see the Matlab code `prob_3_1.m` for the numerical solution to all these Riemann problems.

Problem 3.2 (a numerical linear Riemann solver implementation)

Please see the Matlab code `riemann2x2.m` for an implementation of a Riemann solver for 2x2 systems.

Problem 3.3 (some 3x3 linear Riemann problems)

Part (a): The hyperbolic system to consider is given by $q_t + Aq_x = 0$, with A given by

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of A are found by considering the equation $|A - \lambda I| = 0$ which in this case is

$$\begin{vmatrix} -\lambda & 0 & 4 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 4) = 0\,.$$

This equation has solutions given by $\lambda^1 = -2$, $\lambda^2 = +1$, and $\lambda^3 = +2$. To find the corresponding right eigenvectors r^i, for $i = 1, 2, 3$ we find the nullspace of the matrix $A - \lambda^i I$, for each i. For $i = 1$ this matrix is given by the following system

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 \end{bmatrix} = 0\,,$$

which simplifies under row operations to

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 \end{bmatrix} = 0\,.$$

This expression, demonstrates that $r_1^1 = -2r_3^1$ and $r_2^1 = 0$. Thus a right eigenvector is given by

$$r^1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

The right eigenvector corresponding to λ^2 is obtained by setting $\lambda = 1$ in $A - \lambda I$ and computing the nullspace of this operator. In the same way as above we have the matrix $A - I$ given by

$$\begin{bmatrix} -1 & 0 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \end{bmatrix} = 0\,,$$

which has a nullspace spanned by

$$r^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Finally, by symmetry, for the eigenvector corresponding to λ^3, denoted by r^3 we have

$$r^3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

The R matrix is formed by concatenating the right eigenvectors of A to obtain

$$R = \begin{bmatrix} -2 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The inverse of this matrix can be computed in standard ways and is given by

$$R^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Now the essence of the Riemann solution is to decomposes the jump in state across the initial interface $q_r - q_l$ into a sequence of jumps in the eigenvectors of A. For this problem this decomposition becomes

$$R\alpha = q_r - q_l = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix},$$

so the coefficients of the jumps in the eigenvectors α is given by

$$\alpha = R^{-1} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}.$$

With this information, the Riemann solution for arbitrary time $q(x,t)$ is given by either of two expressions

$$q(x,t) = q_l + \sum_{p:\lambda^p < x/t} \alpha^p r^p = q^r - \sum_{p:\lambda^p \geq x/t} \alpha^p r^p. \tag{59}$$

Using the first expression in the above specified to this problem gives for the value of the left middle state

$$q_{ml} = q_l + \sum_{p:\lambda^p < x/t} \alpha^p r^p = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1/2 \end{bmatrix}$$

As a check on this expression we can compute it using the second expression in Eq. 59 as follows

$$q_{ml} = q_r - \sum_{p:\lambda^p \geq x/t} \alpha^p r^p = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1/2 \end{bmatrix},$$

in agreement with the earlier result. Now to compute the right middle state q_{mr} we again have two possible ways. The first is given by

$$q_{mr} = q_l + \sum_{p:\lambda^p < x/t} \alpha^p r^p = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix},$$

and the second by

$$q_{mr} = q_r - \sum_{p:\lambda^p \geq x/t} \alpha^p r^p = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix},$$

again we have verification of our algebra.

Part (b): For this problem our coefficient matrix A is given by

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

with left and right states given by

$$q_l = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad q_r = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

The solution to linear Riemann problems is given by

$$q(x,t) = q_l + \sum_{p:\lambda_p < x/t} \alpha^p r^p$$

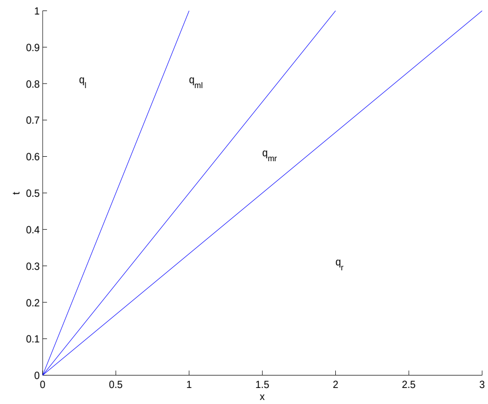

Figure 1: The x-t diagram for the solution of the linear Riemann problem given in problem 3.3 part (b). Please see the Matlab code `prob_3_3_b.m` for the code used to produce these results.

equivalently

$$q(x,t) = q_r - \sum_{p:\lambda_p \geq x/t} \alpha^p r^p \,,$$

both of which involve the eigenvalues and eigenvectors of A. The eigenvalues are given by the solutions λ to $|A - \lambda I| = 0$, or

$$\left| \begin{array}{ccc} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{array} \right| = (1-\lambda)(2-\lambda)(3-\lambda) = 0 \,.$$

This equation has solutions given by $\lambda^1 = 1$, and $\lambda^2 = 2$, and $\lambda^3 = 3$. The corresponding eigenvectors are given by the nullspace to the matrices $A - \lambda^i I$. For λ^1 we have

$$A - I = \left[\begin{array}{ccc} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right] \Rightarrow \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \,.$$

Which gives for the first eigenvector r^1 the expression

$$r^1 = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] \,.$$

For the second eigenvector r^2 we have

$$A - 2I = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives for the second eigenvector r^2 the expression

$$r^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Finally, for the third eigenvector we have

$$A - 3I = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which gives for the third eigenvector r^3 the expression

$$r^3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The matrix (and its inverse) obtained by concatenation of the eigenvectors is given by

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{with} \quad R^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

With these expressions the α vector of coefficients of the initial jump in state in terms of the eigenvector basis is given by

$$\alpha = R^{-1}(q^r - q^l) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Drawing the wedges representing the constant states in the $x - t$ plane we compute the state variables in each one as follows. For the "left" middle state we have

$$
\begin{aligned}
q_{ml} &= q_l + \sum_{p:\lambda_p < x/t} \alpha^p r^p \\
&= q_l + \alpha^1 r^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

For the “right” middle state we have

$$
\begin{aligned}
q_{mr} &= q_l + \sum_{p:\lambda_p < x/t} \alpha^p r^p \\
&= q_l + \alpha^1 r^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} .
\end{aligned}
$$

We can check our calculations for consistency by computing the known right state given all of the previously computed states. Specifically, we have

$$
\begin{aligned}
q_r &= q_l + \sum_{p:\lambda^p < x/t} \alpha^p r^p \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} .
\end{aligned}
$$

which verifies our calculations.

Problem 3.4 (an analytic solution to the linear acoustic equations)

We desire to solve the system $q_t + Aq_x = 0$ with the matrix A and state vector q defined as

$$
A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix} , \quad q = \begin{bmatrix} p \\ u \end{bmatrix} ,
$$

and corresponding initial conditions given by

$$
\mathring{p}(x) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{while} \quad \mathring{u}(x) = 0 \, .
$$

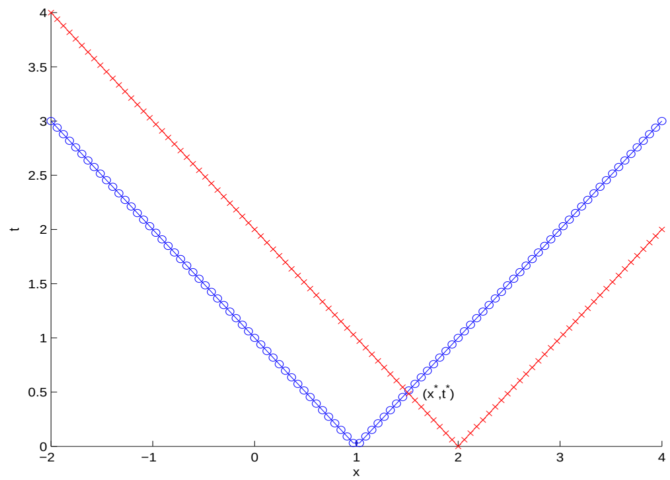

Figure 2: Regions in x-t where the solution u in problem 3.4 changes value.

The eigenvalues and eigenvectors of our matrix A play a fundamental role in the solution and are given by the solution λ to the following equation

$$|A - \lambda I| = \begin{vmatrix} -\lambda & K_0 \\ 1/\rho_0 & -\lambda \end{vmatrix} = 0\,,$$

which has $\lambda^{1,2} = \mp\sqrt{\frac{K_0}{\rho_0}} \equiv \mp c_0$. The eigenvector for $\lambda^1 = -c_0$ are the vector solutions to

$$\begin{bmatrix} c_0 & K_0 \\ 1/\rho_0 & c_0 \end{bmatrix} r^1 = 0\,,$$

or inserting the definition of c_0

$$\begin{bmatrix} \sqrt{\frac{K_0}{\rho_0}} & K_0 \\ 1/\rho_0 & \sqrt{\frac{K_0}{\rho_0}} \end{bmatrix} = 0\,.$$

By multiplying the bottom equation by $\sqrt{\rho_0 K_0}$ we obtain

$$\begin{bmatrix} \sqrt{\frac{K_0}{\rho_0}} & K_0 \\ \sqrt{\frac{K_0}{\rho_0}} & K_0 \end{bmatrix} r^1 = 0\,.$$

Thus we have only one linearly independent equation given by

$$\sqrt{\frac{K_0}{\rho_0}} r_1^1 + K_0 r_2^1 = 0\,,$$

with $K_0 = c_0^2 \rho_0$ the above becomes, $c_0 r_1^1 + c_0^2 \rho_0 r_2^1 = 0$, or $r_1^1 = -c_0 \rho_0 r_2^1$. Resulting in an eigenvector given by

$$r^1 = \begin{bmatrix} -c_0\rho_0 \\ 1 \end{bmatrix}.$$

By symmetry we have that the other eigenvector is given by

$$r^2 = \begin{bmatrix} c_0\rho_0 \\ 1 \end{bmatrix}.$$

The matrix of right eigenvectors becomes

$$R = [r^1 | r^2] = \begin{bmatrix} -c_0\rho_0 & c_0\rho_0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix},$$

by defining $Z_0 = c_0 \rho_0$. The inverse of this matrix R is given by

$$R^{-1} = \frac{1}{2Z_0} \begin{bmatrix} -Z_0 & Z_0 \\ 1 & 1 \end{bmatrix}.$$

With these expressions, the solution for all time is given by a sum of the eigenvectors represented as

$$q(x,t) = w^1(x + c_0 t) r^1 + w^2(x - c_0 t) r^2\,,$$

evaluated at $t = 0$ this becomes

$$q(x,0) = \begin{bmatrix} \mathring{p}(x) \\ \mathring{u}(x) \end{bmatrix} = w^1(x) r^1 + w^2(x) r^2 = R \begin{bmatrix} w^1(x) \\ w^2(x) \end{bmatrix},$$

so the characteristic variables $w^1(x)$, and $w^2(x)$ given by

$$\begin{aligned} \begin{bmatrix} w^1(x) \\ w^2(x) \end{bmatrix} &= R^{-1} \begin{bmatrix} \mathring{p}(x) \\ \mathring{u}(x) \end{bmatrix} \\ &= \frac{1}{2Z_0} \begin{bmatrix} -Z_0 & Z_0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathring{p}(x) \\ \mathring{u}(x) \end{bmatrix} \\ &= \frac{1}{2Z_0} \begin{bmatrix} -\mathring{p}(x) + Z_0 \mathring{u}(x) \\ \mathring{p}(x) + Z_0 \mathring{u}(x) \end{bmatrix}. \end{aligned}$$

Therefore the characteristic variables in terms of the initial conditions are given by

$$\begin{aligned} w^1(x) &= \frac{1}{2Z_0}\left(-\mathring{p}(x) + Z_0\mathring{u}(x)\right) \\ w^2(x) &= \frac{1}{2Z_0}\left(\mathring{p}(x) + Z_0\mathring{u}(x)\right) . \end{aligned}$$

With this expression for the characteristic variables the total solution $q(x,t)$ is given as

$$\begin{aligned} q(x,t) &= \begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix} = R \begin{bmatrix} w^1(x+c_0t) \\ w^2(x-c_0t) \end{bmatrix} \\ &= \begin{bmatrix} -Z_0 & +Z_0 \\ 1 & 1 \end{bmatrix} \frac{1}{2Z_0} \begin{bmatrix} -\mathring{p}(x+c_0t) + Z_0\mathring{u}(x+c_0t) \\ \mathring{p}(x-c_0t) + Z_0\mathring{u}(x-c_0t) \end{bmatrix} \\ &= \frac{1}{2Z_0} \begin{bmatrix} Z_0\left(\mathring{p}(x+c_0t) + \mathring{p}(x-c_0t)\right) - Z_0^2\left(\mathring{u}(x+c_0t) - \mathring{u}(x-c_0t)\right) \\ -\mathring{p}(x+c_0t) + \mathring{p}(x-c_0t) + Z_0\left(\mathring{u}(x+c_0t) + \mathring{u}(x-c_0t)\right) \end{bmatrix} . \end{aligned}$$

Now simply substituting the arguments of $\mathring{p}$ and $\mathring{u}$, (namely $x + c_0t$ and $x - c_0t$) into the expressions above and using the specific initial conditions provided with this problem we have for $p(x,t)$ the following expression

$$p(x,t) = \frac{1}{2}\begin{cases} 1 & 1 \leq x + c_0t \leq 2 \\ 0 & \text{otherwise} \end{cases} + \frac{1}{2}\begin{cases} 1 & 1 \leq x - c_0t \leq 2 \\ 0 & \text{otherwise} \end{cases} . \tag{60}$$

This can be simplified by considering in the x-t space the location of the lines $1 = x \pm c_0t$ and $2 = x \pm c_0t$. In figure 2 we have drawn these lines for $c_0 = 1$. See the Matlab script `prob_3_4.m` for the commands to produce this plot. Then depending on which region of this plot our (x,t) point falls the various components of Eq. 61 will either contribute or not. Obviously one location in time is when the two lines $x + c_0t = 2$ and $x - c_0t = 1$ intersect for after that time the outgoing characteristics don't overlap. This time t^* is then the solution to

$$2 - c_0t^* = 1 + c_0t^*$$

or $t^* = \frac{1}{2c_0}$. For all times $t < t^*$ for each region in figure 2 we can evaluate what components of Eq. 61 contribute to the total solution for $p(x,t)$. When this is done we have that $p(x,t)$ is given by

$$p(x,t) = \begin{cases} 0 & x < 1 - c_0 t \\ \frac{1}{2} & 1 - c_0 t < x < 1 + c_0 t \\ 1 & 1 + c_0 t < x < 2 - c_0 t \\ \frac{1}{2} & 2 - c_0 t < x < 2 + c_0 t \\ 0 & x > 2 + c_0 t \end{cases}$$

Now if $t > t^*$ we can do the same thing and obtain

$$p(x,t) = \begin{cases} 0 & x < 1 - c_0 t \\ \frac{1}{2} & 1 - c_0 t < x < 2 - c_0 t \\ 0 & 2 - c_0 t < x < 1 + c_0 t \\ \frac{1}{2} & 1 + c_0 t < x < 2 + c_0 t \\ 0 & x > 2 + c_0 t \end{cases}$$

In the same way, for all t we have $u(x,t)$ given by

$$u(x,t) = \frac{1}{2Z_0} \begin{cases} -1 & 1 \le x + c_0 t \le 2 \\ 0 & \text{otherwise} \end{cases} + \frac{1}{2Z_0} \begin{cases} 1 & 1 \le x - c_0 t \le 2 \\ 0 & \text{otherwise} \end{cases} . \quad (61)$$

When $t < t^*$ the above can be simplified and we obtain

$$u(x,t) = \begin{cases} 0 & x < 1 - c_0 t \\ -\frac{1}{4Z_0} & 1 - c_0 t < x < 1 + c_0 t \\ 0 & 1 + c_0 t < x < 1 + c_0 t \\ \frac{1}{4Z_0} & 2 - c_0 t < x < 2 + c_0 t \\ 0 & x > 2 + c_0 t \end{cases}$$

Where for $t > t^*$ we have

$$u(x,t) = \begin{cases} 0 & x < 1 - c_0 t \\ -\frac{1}{4Z_0} & 1 - c_0 t < x < 2 - c_0 t \\ 0 & 2 - c_0 t < x < 1 + c_0 t \\ \frac{1}{4Z_0} & 1 + c_0 t < x < 2 + c_0 t \\ 0 & x > 2 + c_0 t \end{cases}$$

Chapter 4 (Finite Volume Methods)

Problem 4.1 (the matrices A^+ and A^- for the acoustic equations)

The acoustic equations are given by LeVeque Eq. 2.50 which is a linear hyperbolic system of the form $q_t + Aq_x = 0$ with the matrix A and state vector

q defined as

$$A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}, \quad q = \begin{bmatrix} p \\ u \end{bmatrix}.$$

Following the same procedure as in Problem 3.4 we have the eigenvalues and eigenvectors of A given by

$$\begin{aligned} \lambda^1 &= u_0 - c_0 \quad \text{with} \quad r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} \\ \lambda^2 &= u_0 + c_0 \quad \text{with} \quad r^2 = \begin{bmatrix} +\rho_0 c_0 \\ 1 \end{bmatrix}. \end{aligned}$$

Below we may use the definition of the impedance Z_0 given by $Z_0 \equiv \rho_0 c_0$.

Part (a): To determine A^+ and A^- we first compute the eigenvector factorization of A, i.e. $A = R\Lambda R^{-1}$. Our matrix R (and its inverse) is then given by

$$R = \begin{bmatrix} -Z_0 & +Z_0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad R^{-1} = \frac{1}{2Z_0} \begin{bmatrix} 1 & -Z_0 \\ -1 & -Z_0 \end{bmatrix},$$

and our diagonal matrix of eigenvalues Λ is given by

$$\Lambda = \begin{bmatrix} u_0 - c_0 & 0 \\ 0 & u_0 + c_0 \end{bmatrix}$$

Assume for the time being that we are in the case of subsonic flow $u_0 < c_0$, then $u_0 - c_0 < 0$ so we have

$$\Lambda^- = \begin{bmatrix} \min(u_0 - c_0, 0) & 0 \\ 0 & \min(u_0 + c_0, 0) \end{bmatrix} = \begin{bmatrix} u_0 - c_0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\Lambda^+ = \begin{bmatrix} \max(u_0 - c_0, 0) & 0 \\ 0 & \max(u_0 + c_0, 0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & u_0 + c_0 \end{bmatrix}.$$

so we can now assemble A^- and A^+ from the component parts Λ^-, Λ^+, R, and R^{-1}. We have for A^-

$$A^- = R\Lambda^- R^{-1} = \frac{u_0 - c_0}{2Z_0} \begin{bmatrix} Z_0 & -Z_0^2 \\ -1 & Z_0 \end{bmatrix},$$

and for A^+

$$A^+ = R\Lambda^+ R^{-1} = \frac{u_0 + c_0}{2Z_0} \begin{bmatrix} Z_0 & Z_0^2 \\ 1 & Z_0 \end{bmatrix}.$$

If we instead assume this flow is supersonic i.e. $u_0 > c_0$ or $u_0 - c_0 > 0$, we then have for Λ^- and Λ^+ the following

$$\Lambda^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Lambda^+ = \begin{bmatrix} u_0 - c_0 & 0 \\ 0 & u_0 + c_0 \end{bmatrix},$$

so reconstructing A^- and A^+ gives

$$A^- = 0 \quad \text{and} \quad A^+ = A\,.$$

We can explicitly check that in this case $A^+ = A$ by computing the matrix $A^+ = R\Lambda^+ R^{-1}$ as follows

$$\begin{aligned} A^+ &= R\Lambda^+ R^{-1} \\ &= \begin{bmatrix} -Z_0 & +Z_0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_0 - c_0 & 0 \\ 0 & u_0 + c_0 \end{bmatrix} \frac{1}{2Z_0} \begin{bmatrix} -1 & +Z_0 \\ 1 & Z_0 \end{bmatrix} \\ &= \frac{1}{2Z_0} \begin{bmatrix} -Z_0(u_0 - c_0) & Z_0(u_0 + c_0) \\ (u_0 - c_0) & (u_0 + c_0) \end{bmatrix} \begin{bmatrix} -1 & +Z_0 \\ 1 & Z_0 \end{bmatrix} \\ &= \frac{1}{2Z_0} \begin{bmatrix} Z_0(2u_0) & Z_0^2(2c_0) \\ 2c_0 & Z_0(2u_0) \end{bmatrix} \\ &= \begin{bmatrix} u_0 & c_0 Z_0 \\ \frac{c_0}{Z_0} & u_0 \end{bmatrix}. \end{aligned}$$

Now since $c_0 = \sqrt{\frac{K_0}{\rho_0}}$ and $Z_0 = \rho_0 c_0$ we have the following simplifications of the expressions c_0/Z_0 and $c_0 Z_0$:

$$\begin{aligned} \frac{c_0}{Z_0} &= \frac{c_0}{\rho_0 c_0} = \frac{1}{\rho_0} \\ c_0 Z_0 &= c_0 \rho_0 c_0 = \rho_0 c_0^2 = \rho_0 \left(\frac{K_0}{\rho_0} \right)^2 = \frac{\rho_0}{\rho_0} K_0 = K_0\,. \end{aligned}$$

Together with these modifications we see explicitly that $A^+ = A$ in the supersonic case.

Part (b): Fow systems we have that $W^1_{i-1/2} = \alpha^1 r^1$, and $W^2_{i-1/2} = \alpha^2 r^2$, where the α's are determined from the jump in state across $x_{i-1/2}$. To compute these coefficients we must solve the following linear system for the vector of α

$$R\alpha = Q_i - Q_{i-1}\,,$$

thus formally α is given by

$$\alpha = R^{-1}(Q_i - Q_{i-1}) \,.$$

For the linear acoustic problem considered here the system above is explicitly given by

$$\begin{aligned} \begin{bmatrix} \alpha^1_{i-1/2} \\ \alpha^2_{i-1/2} \end{bmatrix} &= \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix} \begin{bmatrix} p_i - p_{i-1} \\ u_i - u_{i-1} \end{bmatrix} \\ &= \frac{1}{2Z_0} \begin{bmatrix} -(p_i - p_{i-1}) + Z_0(u_i - u_{i-1}) \\ +(p_i - p_{i-1}) + Z_0(u_i - u_{i-1}) \end{bmatrix} \end{aligned}$$

So with these coefficients the waves (for an arbitrary jump) are given by

$$W^1_{i-1/2} = \alpha^1_{i-1/2} r^1 = \frac{1}{2Z_0} \left(-(p_i - p_{i-1}) + Z_0(u_i - u_{i-1}) \right) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

and

$$W^2_{i-1/2} = \alpha^2_{i-1/2} r^2 = \frac{1}{2Z_0} \left(+(p_i - p_{i-1}) + Z_0(u_i - u_{i-1}) \right) \begin{bmatrix} +Z_0 \\ 1 \end{bmatrix} .$$

Problem 4.2 (the unit CFL condition)

The first order upwind method for $q_t + \bar{u} q_x = 0$, is given by LeVeque Eq. 4.25 or

$$Q^{n+1}_i = Q^n_i - \frac{\bar{u}\Delta t}{\Delta x}(Q^n_i - Q^n_{i-1})$$

if we assume a unit CFL condition i.e. that $\frac{\bar{u}\Delta t}{\Delta x} = 1$, then the above reduces to

$$Q^{n+1}_i = Q^n_{i-1}$$

Part (b): The Lax-Friedrichs method, LeVeque Eq. 4.20 is given by

$$Q^{n+1}_i = \frac{1}{2}(Q^n_{i-1} + Q^n_{i+1}) - \frac{\Delta t}{2\Delta x}(f(Q^n_{i+1}) - f(Q^n_{i-1})) \,.$$

For the advection equation $q_t + \bar{u} q_x = 0$ we have $f(q) = \bar{u} q$, therefore Lax-Fredrich's method in this case becomes

$$Q^{n+1}_i = \frac{1}{2}(Q^n_{i-1} + Q^n_{i+1}) - \frac{\bar{u}\Delta t}{2\Delta x}(Q^n_{i+1} - Q^n_{i-1}) \,.$$

With a unit CFL we have $\bar{u}\Delta t/\Delta x = 1$ which in the above gives

$$Q_i^{n+1} = \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{1}{2}(Q_{i+1}^n - Q_{i-1}^n) = \frac{1}{2}(2Q_{i-1}^n) = Q_{i-1}^n\,,$$

and thus the Lax-Friedrich method satisfies the unit CFL condition. For the Lax-Wendroff method, we update Q_i^{n+1} with

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(f(Q_{i+1/2}^{n+1/2}) - f(Q_{i-1/2}^{n+1/2})\right)\,,$$

with the state at the half time step $Q_{i-1/2}^{n+1/2}$ given by

$$Q_{i-1/2}^{n+1/2} = \frac{1}{2}(Q_{i-1}^n + Q_i^n) - \frac{\Delta t}{2\Delta x}(f(Q_i^n) - f(Q_{i-1}^n))\,.$$

Specifying to the scalar advection equation we have a flux given by $f(q) = \bar{u}q$, and evaluating the above under the unit CFL condition gives for the half timestep state $Q_{i-1/2}^{n+1/2}$ the following

$$Q_{i-1/2}^{n+1/2} = \frac{1}{2}(Q_{i-1}^n + Q_i^n) - \frac{1}{2}(Q_i^n - Q_{i-1}^n) = Q_{i-1}^n$$

so when used in the full timestep update equation we have

$$Q_i^{n+1} = Q_i^n - (Q_{i+1/2}^{n+1/2} - Q_{i-1/2}^{n+1/2}) = Q_i^n - (Q_i^n - Q_{i-1}^n) = Q_{i-1}^n\,.$$

Showing that the Lax-Wendroff method satisfies the unit CFL condition.
Part (c): The constant coefficient acoustic equations, LeVeque Eq. 2.50, with $u_0 = 0$ are given by

$$q_t + \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix} q_x = 0 \quad \text{with} \quad q = \begin{bmatrix} p \\ u \end{bmatrix}\,.$$

Now Godunov's method is a flux differencing method, defined by LeVeque Eq. 4.4 and given by

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

with a flux given by

$$F_{i-1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(Q_{i-1}^n, Q_i^n))dt\,,$$

where $q(Q^n_{i-1}, Q^n_i)$ is to be evaluated along the interface at $x = x_{i-1/2}$. Note that for linear constant coefficient problems (as given here), the Riemann problem can be solved and the integral above can be computed exactly. Specifically with $f(q) = Aq$, we have a numeric flux given by

$$\begin{aligned} F^n_{i-1/2} &= A\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} q(Q^n_{i-1}, Q^n_i)dt \\ &= Aq(Q^n_{i-1}, Q^n_i) \\ &= A(Q^n_i - \sum_{p:\lambda^p>0} \alpha^p_{i-1/2} r^p_{i-1/2}) \\ &= AQ^n_i - \sum_{p:\lambda^p>0} \lambda^p \alpha^p_{i-1/2} r^p_{i-1/2}\,. \end{aligned}$$

Where $\alpha^p_{i-1/2}$ comes from decomposing the jump in state at the $x = x_{i-1/2}$ interface into characteristic waves. From previous problems the coefficients α^p are given by solving the Riemann problem and are given by

$$\begin{aligned} \alpha^n_{i-1/2} &= R^{-1}(Q^n_i - Q^n_{i-1}) = \frac{1}{2Z_0}\begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix}\begin{bmatrix} p^n_i - p^n_{i-1} \\ u^n_i - u^n_{i-1} \end{bmatrix} \\ &= \frac{1}{2Z_0}\begin{bmatrix} -(p^n_i - p^n_{i-1}) + Z_0(u^n_i - u^n_{i-1}) \\ (p^n_i - p^n_{i-1}) + Z_0(u^n_i - u^n_{i-1}) \end{bmatrix}. \end{aligned}$$

Because with the constant coefficient acoustic equations the eigenvalues are constant with known sign we have $\lambda^1 = -c_0 < 0$, and $\lambda^2 = +c_0 > 0$, we can explicitly evaluate the summation in the numerical flux $F^n_{i-1/2}$ giving

$$F^n_{i-1/2} = AQ^n_i - \lambda^2\alpha^2_{i-1/2}r^2\,.$$

A part of this flux is given by $\alpha^2_{i-1/2}r^2$ or

$$\alpha^2_{i-1/2}r^2 = \frac{1}{2Z_0}\big((p^n_i - p^n_{i-1}) + Z_0(u^n_i - u^n_{i-1})\big)\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

Combined with AQ^n_i (and $\lambda^2 = c_0$) we have a numerical flux given by

$$\begin{aligned} F^n_{i-1/2} &= \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}\begin{bmatrix} p^n_i \\ u^n_i \end{bmatrix} - \frac{c_0}{2Z_0}\big((p^n_i - p^n_{i-1}) + Z_0(u^n_i - u^n_{i-1})\big)\begin{bmatrix} Z_0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -c_0(p^n_i - p^n_{i-1}) + K_0(u^n_i + u^n_{i-1}) \\ (1/\rho_0)(p^n_i + p^n_{i-1}) - c_0(u^n_i - u^n_{i-1}) \end{bmatrix}. \end{aligned}$$

The required flux difference found in Godunov's method then evaluates to

$$F^n_{i+1/2} - F^n_{i-1/2} = \frac{1}{2}\begin{bmatrix} -c_0(p^n_{i+1} - 2p^n_i + p^n_{i-1}) + K_0(u^n_{i+1} - u^n_{i-1}) \\ (1/\rho_0)(p^n_{i+1} - p^n_{i-1}) - c_0(u^n_{i+1} - 2u^n_i + u^n_{i-1}) \end{bmatrix}.$$

Thus Godunov's method in these variables is given by

$$\begin{bmatrix} p^{n+1}_i \\ u^{n+1}_i \end{bmatrix} = \begin{bmatrix} p^n_i \\ u^n_i \end{bmatrix} - \frac{\Delta t}{2\Delta x}\begin{bmatrix} -c_0(p^n_{i+1} - 2p^n_i + p^n_{i-1}) + K_0(u^n_{i+1} - u^n_{i-1}) \\ (1/\rho_0)(p^n_{i+1} - p^n_{i-1}) - c_0(u^n_{i+1} - 2u^n_i + u^n_{i-1}) \end{bmatrix}.$$

Imposing a unit CFL condition $c_0\Delta t/\Delta x = 1$ we have from the above that the pressure and velocity are updated according to

$$\begin{bmatrix} p^{n+1}_i \\ u^{n+1}_i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} p^n_{i+1} + p^n_{i-1} - \rho_0 c_0(u^n_{i+1} - u^n_{i-1}) \\ -(p^n_{i+1} - p^n_{i-1})/(\rho_0 c_0) + (u^n_{i+1} + u^n_{i-1}) \end{bmatrix}.$$

where we have used the fact that $c_0 = \sqrt{\frac{K_0}{\rho_0}}$ in deriving the above.

To show that the result above for (p^{n+1}_i, u^{n+1}_i) is the *exact* solution as obtained by characteristic theory we consider what a characteristic update of the state Q^{n+1}_i would require. To update the state at Q^{n+1}_i using the method of characteristics we would propagate each characteristic back to the time t^n, where we would use the characteristic decomposition of the states to update the value at Q^{n+1}_i. Since we are propagating backwards using a timestep Δt such that the CFL number is *exactly* one, the left and right characteristics intersect the x-axis at the points x_{i-1} and x_{i+1} exactly, where the states are given by Q^n_{i-1} and Q^n_{i+1} respectively. By propagating the *right* going characteristic backwards to x_{i-1} we should compute the right going characteristic variable. This means we first decompose the state Q^n_{i-1} as

$$Q^n_{i-1} = \alpha^1 r^1 + \alpha^2 r^2 .$$

For the acoustic equations the coefficients α are given by

$$\begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix} = \frac{1}{2Z_0}\begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix}\begin{bmatrix} p^n_{i-1} \\ u^n_{i-1} \end{bmatrix}$$

so the right going characteristic variable used to update the state at Q^{n+1}_i is given by

$$\alpha^2 r^2 = \frac{1}{2Z_0}(p^n_{i-1} + Z_0 u^n_{i-1})\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

Propagating the *left* going characteristic backwards to x_{i+1} we should compute the left going characteristic variable from the state at Q_{i+1}^n. This means that we decompose the state Q_{i+1}^n as

$$Q_{i+1}^n = \beta^1 r^1 + \beta^2 r^2 \,.$$

For the acoustic equations the coefficients β are given by

$$\begin{bmatrix} \beta^1 \\ \beta^2 \end{bmatrix} = \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix} \begin{bmatrix} p_{i+1}^n \\ u_{i+1}^n \end{bmatrix}$$

so the left going characteristic variable to update the state at Q_i^{n+1} is given by

$$\beta^1 r^1 = \frac{1}{2Z_0}(-p_{i+1}^n + Z_0 u_{i+1}^n) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} .$$

At this point the characteristic solution for Q_i^{n+1} is given by the superposition of these two characteristic updates or

$$\begin{aligned} Q_i^{n+1} &= \alpha^2 r^2 + \beta^1 r^1 \\ &= \frac{1}{2Z_0}(p_{i-1}^n + Z_0 u_{i-1}^n) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix} + \frac{1}{2Z_0}(-p_{i+1}^n + Z_0 u_{i+1}^n) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} p_{i+1}^n + p_{i-1}^n - Z_0(u_{i+1}^n - u_{i-1}^n) \\ (p_{i-1}^n - p_{i+1}^n)/Z_0 + (u_{i+1}^n + u_{i-1}^n) \end{bmatrix} \end{aligned}$$

This is *exactly* the same update equation derived using Godunov's method and a unit CFL condition.

Part (d): In the case when $u_0 \neq 0$, the two wave emanating from a given interface travel at two speeds with *different* magnitudes, i.e. $u_0 - c_0$ and $u_0 + c_0$. Thus there is no hope that in taking a single timestep we will be able to propagate all of the waves exactly as is done when all waves move with magnitude c_0. Thus it is not possible, to obtain results similar to the above in the case $u_0 \neq 0$.

Problem 4.3 (large timestep wave propagation algorithms)

Part (a): When $\Delta x \leq \bar{u}\Delta t \leq 2\Delta x$, the waves that originate from $x_{i-1/2}$ propagate thought the neighboring interface at $x_{i+1/2}$ and into the next cell. Thus the interface that found in the cell $\mathcal{C}_i$ at time t^{n+1} originated at the interface $x_{i-3/2}$ at time t^n. Developing the REA method as in the text we have that the cell average in cell $\mathcal{C}_i$ at the time t^{n+1} would be given by

$$
\begin{aligned}
Q_i^{n+1} &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x, t_{n+1})dx \\
&= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{\xi} \tilde{q}^n(x, t_{n+1})dx + \frac{1}{\Delta x}\int_{\xi}^{x_{i+1/2}} \tilde{q}^n(x, t_{n+1})dx \,.
\end{aligned}
$$

Where ξ (the location of the discontinuity in cell $\mathcal{C}_i$ at time t^{n+1}) is located at

$$\xi = x_{i-3/2} + \bar{u}(t^n - t^{n+1}) = x_{i-3/2} + \bar{u}\Delta t \,.$$

The state to the left of this discontinuity ξ is given by Q_{i-2}^n, while the state to the right of this discontinuity is given by Q_{i-1}^n so we have using the REA formulation that

$$
\begin{aligned}
Q_i^{n+1} &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i-3/2}+\bar{u}\Delta t} Q_{i-2}^n dx + \frac{1}{\Delta x}\int_{x_{i-3/2}-\bar{u}\Delta t}^{x_{i+1/2}} Q_{i-1}^n dx \\
&= \frac{Q_{i-2}^n}{\Delta x}(-\Delta x + \bar{u}\Delta t) + \frac{Q_{i-1}^n}{\Delta x}(2\Delta x - \bar{u}\Delta t) \\
&= 2Q_{i-1}^n - \frac{\bar{u}\Delta t}{\Delta x}Q_{i-1}^n - Q_{i-2}^n + \frac{\bar{u}\Delta t}{\Delta x}Q_{i-2}^n \,.
\end{aligned}
$$

From the above we have that

$$
\begin{aligned}
Q_i^{n+1} &= Q_{i-1}^n + \left(1 - \frac{\bar{u}\Delta t}{\Delta x}\right) Q_{i-1}^n + \left(-1 + \frac{\bar{u}\Delta t}{\Delta x}\right) Q_{i-2}^n \\
&= Q_{i-1}^n - \left(\frac{\bar{u}\Delta t}{\Delta x} - 1\right)(Q_{i-1}^n - Q_{i-2}^n) \,,
\end{aligned}
$$

which is LeVeque Eq. 4.64 and the desired expression.

Part (b): From Figure 4.4 (a), we see that the state at (x_i, t^{n+1}) is obtained by propagating backwards along characteristics to the time t^n. If the characteristics intersect the x-axis at a point *between* two grid points the state is computed via. interpolation. For the large timestep case considered here, the point we intersect the x-axis (at time t^n) is at

$$\xi = x_i - \bar{u}\Delta t \,,$$

which for the large timestep we are considering here falls between the states Q^n_{i-2} and Q^n_{i-1}. Interpolating between these two points we get for Q^{n+1}_i the following

$$Q^{n+1}_i = \left(\frac{\xi - x_{i-2}}{\Delta x}\right) Q^n_{i-1} + \left(\frac{x_{i-1} - \xi}{\Delta x}\right) Q^n_{i-2}\,.$$

With some simplification becomes

$$Q^{n+1}_i = Q^n_{i-1} - \left(\frac{\bar{u}\Delta t}{\Delta x} - 1\right)(Q^n_{i-1} - Q^n_{i-2})\,,$$

which is the same as the result earlier.

Part (c): Now if $\bar{u}\Delta t/\Delta x = 1$ then from the above expression we see that $Q^{n+1}_i = Q^n_{i-1}$ which propagation of the state from the left cell into the current cell and is the exact solution. If $\bar{u}\Delta t/\Delta x = 2$ we have that the above gives

$$Q^{n+1}_i = Q^n_{i-1} - (Q^n_{i-1} - Q^n_{i-2}) = Q^n_{i-2}\,,$$

which is also exact for this larger timestep.

Part (d): For the CFL condition to hold we must have the numerical domain of dependence contain the physical domain of dependence. We can see from the numerical method used that the point Q^{n+1}_i has a numerical domain of dependence that includes the points Q^n_{i-1} and Q^n_{i-2}. Specifically for the range of CFL numbers given by

$$1 \le \frac{\bar{u}\Delta t}{\Delta x} \le 2\,,$$

the numerical domain of dependence of the point X is given by the points x defined by

$$X - \frac{T}{\frac{\Delta t}{2\Delta x}} \le x \le X - \frac{T}{\frac{\Delta t}{\Delta x}}\,.$$

Now since the *mathematical* domain of dependence of the point (X,T) is the single point $X - \bar{u}T$, so the CFL condition would require that

$$X - \frac{T}{\frac{\Delta t}{2\Delta x}} \le X - \bar{u}T \le X - \frac{T}{\frac{\Delta t}{\Delta x}}$$

which can be simplified to

$$\frac{-2\Delta x}{\Delta t} \le -\bar{u} \le \frac{-\Delta x}{\Delta t}$$

or

$$1 \leq \frac{\Delta t \bar{u}}{\Delta x} \leq 2\,,$$

for a CFL condition.
Part (e): If $2\Delta x \leq \bar{u}\Delta t \leq 3\Delta x$ we would apply the same methods earlier on in this problem. For example, by backtracking characteristics to the x axis we see that because of the large timestep taken, the characteristic from the point (x_i, t^{n+1}) will pass between the points x_{i-3} and x_{i-2}. We can obtain a numerical method by interpolating between the two unknowns at those points namely Q_{i-3}^n and Q_{i-2}^n. Specifically we will construct Q_i^{n+1} from

$$Q_i^{n+1} = \left(\frac{\xi - x_{i-3}}{\Delta x}\right) Q_{i-2}^n + \left(\frac{x_{i-2} - \xi}{\Delta x}\right) Q_{i-3}^n\,.$$

where ξ is the point in between the two cell centers i.e.

$$\xi = x_i - \bar{u}\Delta t\,,$$

With some simplification the above becomes

$$Q_i^{n+1} = \left(3 - \frac{\bar{u}\Delta t}{\Delta x}\right) Q_{i-2}^n + \left(-2 + \frac{\bar{u}\Delta t}{\Delta x}\right) Q_{i-3}^n\,.$$

Chapter 5 (Introduction to the CLAWPACK Software)

Problem 5.1 (a comparison between first and second order methods)

To have CLAWPACK compute with a first order method, simply change the line

```
2                    method(2)    = order
```

in the `claw1ez.data` file to read

```
1                    method(2)    = order
```

and execute the `xclaw` binary. Visual results output from from this can be found on the web page.

Problem 5.2 (Instability with a Courant number above 1)

To have CLAWPACK compute with a Courant number of 1.1 in the `claw1ez.data`, simply change the two lines

```
1.0d0              cflv(1)     = max allowable Courant number
0.9d0              cflv(2)     = desired Courant number
```

to the following

```
1.1d0              cflv(1)     = max allowable Courant number
1.1d0              cflv(2)     = desired Courant number
```

and execute the `xclaw` binary. Visual results output from from this can be found on the web page, where one can see the instability that results.

Problem 5.3 (A simple wave linear acoustics)

Wave that propagates entirely in one direction are denoted simple waves and thus this problem is asking us to find a simple wave for the linear acoustic equations. From the eigenvector decomposition of the primitive variables at $t = 0$ of

$$\begin{bmatrix} p(x,0) \\ u(x,0) \end{bmatrix} = w^1(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + w^2(x) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix},$$

we will have a solution (for all time) given by

$$\begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix} = w^1(x + c_0 t) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + w^2(x - c_0 t) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix},$$

and thus no propagation in the right direction if initially the function $w^2(x) \equiv 0$. Thus we should take the given functional form for $p(x,0)$ and convert the initial conditions into something that looks like

$$\begin{bmatrix} p(x,0) \\ u(x,0) \end{bmatrix} = w^1(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}.$$

We can proceed by factoring out $p(x,0)$ as follows

$$\begin{bmatrix} p(x,0) \\ u(x,0) \end{bmatrix} = -\frac{p(x,0)}{Z_0} \begin{bmatrix} -Z_0 \\ -Z_0 \left(\frac{u(x,0)}{p(x,0)} \right) \end{bmatrix}.$$

which will have the correct form if the expression $-Z_0 \left(\frac{u(x,0)}{p(x,0)} \right)$ is set equal to one. This implys that our initial condition for the velocity be chosen as

$$u(x,0) = -\frac{p(x,0)}{Z_0}\,.$$

This can be implemented in the CLAWPACK software by changing the procedure `qinit.f`. Specifically, in this set of problems the line

```
q(i,2) = 0.d0
```

is replaced with the following

```
q(i,2) = -q(i,1)/(rho*cc)
```

Results with that change can be found on the web sight.

Chapter 6 (High resolution methods)

Problem 6.1 (the REA method for the scalar advection equation)

LeVeque Eq. 6.13 is given by

$$Q_i^{n+1} = \frac{\bar{u}\Delta t}{\Delta x}\left(Q_{i-1}^n + \frac{1}{2}(\Delta x - \bar{u}\Delta t)\sigma_{i-1}^n \right) + \left(1 - \frac{\bar{u}\Delta t}{\Delta x} \right)(Q_i^n - \frac{1}{2}\bar{u}\Delta t \sigma_i^n)$$

To derive this result using the REA framework, first define a piecewise reconstructed linear function given in each cell by

$$\tilde{q}^n(x,t) = Q_i^n + \sigma_i^n(x - x_i)\,.$$

The second step in the REA framework is to propagate this solution forward to the new time t^{n+1}. For the constant coefficient scalar advection equation the reconstructed solution at time t^{n+1} or $\tilde{q}^n(x, t_{n+1})$ is given by

$$\tilde{q}^n(x, t_{n+1}) = \tilde{q}^n(x - \Delta t\bar{u}, t_n)\,.$$

The final step in the REA framework is to calculate the average over the cell $\mathcal{C}_i$, which is defined by the interval $(x_{i-1/2}, x_{i+1/2})$. Performing this averaging we have

$$\begin{aligned} Q_i^{n+1} &\equiv \frac{1}{\Delta x}\int_{C_i} \tilde{q}(x, t_{n+1})dx \\ &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x - \bar{u}\Delta t, t_n)dx \end{aligned}$$

Performing the required algebra and recognizing that at time t^n the state changes value at $x_{i-1/2}$ so we will need to break any integrals at that point we have

$$\begin{aligned}
Q_i^{n+1} &= \frac{1}{\Delta x}\int_{x_{i-1/2}-\bar{u}\Delta t}^{x_{i+1/2}-\bar{u}\Delta t} \tilde{q}^n(\xi, t_n)d\xi \\
&= \frac{1}{\Delta x}\int_{x_{i-1/2}-\bar{u}\Delta t}^{x_{i-1/2}} \tilde{q}^n(\xi, t_n)d\xi + \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}-\bar{u}\Delta t} \tilde{q}^n(\xi, t_n)d\xi \\
&= \frac{1}{\Delta x}\int_{x_{i-1/2}-\bar{u}\Delta t}^{x_{i-1/2}} \left(Q_{i-1}^n - \sigma_{i-1}^n(\xi - x_{i-1})\right) d\xi \\
&+ \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}-\bar{u}\Delta t} \left(Q_i^n + \sigma_i^n(\xi - x_i)\right) d\xi \,.
\end{aligned}$$

When the integration is performed we obtain

$$Q_i^{n+1} = \frac{\bar{u}\Delta t}{\Delta x}(Q_{i-1}^n + \frac{1}{2}(\Delta x - \bar{u}\Delta t)\sigma_{i-1}^n) + (1 - \frac{\bar{u}\Delta t}{\Delta x})(Q_i^n - \frac{\sigma_i^n}{2}\bar{u}\Delta t)\,,$$

which is the desired result.

Problem 6.2 (examples calculating the total variation)

The total variation can be defined in one of several ways. For a grid function (defined at the grid nodes) we have

$$\mathrm{TV}(Q) = \sum_{i=-\infty}^{\infty} |Q_i - Q_{i-1}|\,.$$

For more general functions one can use any of the following definitions

$$\begin{aligned}
\mathrm{TV}(q) &= \sup_{\xi} \sum_{j=1}^{N} |q(\xi_j) - q(\xi_{j-1})| \\
\mathrm{TV}(q) &= \lim_{\epsilon\to 0}\sup \frac{1}{\epsilon}\int_{-\infty}^{\infty} |q(x) - q(x-\epsilon)| dx \\
\mathrm{TV}(q) &= \int_{-\infty}^{\infty} |q'(x)| dx
\end{aligned}$$

Part (a): For the function $q(x)$ defined by

$$q(x) = \begin{cases} 1 & x < 0 \\ \sin(\pi x) & 0 \leq x \leq 3 \\ 2 & x > 3 \end{cases}$$

we compute the total variation using

$$\begin{aligned}
\mathrm{TV}(q) &= \lim_{\epsilon \to 0} \sup \frac{1}{\epsilon} \int_{-\infty}^{\infty} |q(x) - q(x-\epsilon)| dx \\
&= \lim_{\epsilon \to 0} \sup \frac{1}{\epsilon} \left(\int_{-\eta_1}^{\eta_1} |q(x) - q(x-\epsilon)| dx \right. \\
&+ \int_{\eta_1}^{3-\eta_2} |q(x) - q(x-\epsilon)| dx \\
&+ \left. \int_{3-\eta_2}^{\eta_2} |q(x) - q(x-\epsilon)| dx \right) \\
&= 1 + \int_0^3 |q'(x)| dx + 2 \\
&= 3 + \pi \int_0^{1/2} \cos(\pi x) dx + \pi \int_{1/2}^{3/2} - \cos(\pi x) dx \\
&+ \pi \int_{3/2}^{5/2} \cos(\pi x) dx + \pi \int_{5/2}^{3} - \cos(\pi x) dx \\
&= 10
\end{aligned}$$

Part (b): For the function $q(x)$ defined by

$$q(x) = \begin{cases} 1 & x < 0 \quad \text{or} \quad x = 3 \\ 1 & 0 \leq x \leq 1 \quad \text{or} \quad 2 \leq x \leq 3 \\ -1 & 1 < x < 2 \\ 2 & x > 3 \end{cases}$$

Using the definition of the total variation is given by

$$\mathrm{TV}(q) = \sup_{\xi} \sum_{j=1}^{N} |q(\xi_j) - q(\xi_{j-1})| \, ,$$

we have

$$\mathrm{TV}(q) \;=\; |q(1^+) - q(1^-)| + |q(2^+) - q(2^-)| + |q(3^+) - q(3^-)|$$

$$
\begin{aligned}
&= |-1-1| + |1+1| + |2-1| \\
&= 2+2+1 = 5
\end{aligned}
$$

Problems 6.3 (any TVD method is also monotonicity-preserving)

We will prove the contrapositve of the given statement, i.e. that if our numerical method is not montonicity-preserving then it it not TVD.

Problem 6.4 (averaging is a TVD process)

LeVeque Eq. 6.25 claims that $\text{TV}(Q^{n+1}) \le \text{TV}(\tilde{q}^n(\cdot, t_{n+1}))$, with the cell values at time t^{n+1} obtained by averaging the evolved profile $\tilde{q}^n$ as

$$
Q_i^{n+1} \equiv \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(\xi, t_{n+1}) d\xi \,.
$$

Thus cell averaging can only decrease the total variation and is a total variation decreasing operation. To prove this, consider a arbitrary function $q(x)$ and define its cell averages by the standard formula

$$
Q_i = \frac{1}{\Delta x} \int_{C_i} q(x) dx \,,
$$

then our claim for $q(x)$ is that $\text{TV}(Q_i) \le \text{TV}(q)$. Where the grid function Q_i has a total variation defined by the standard formula

$$
\text{TV}(Q_i) = \sum_{i=-\infty}^{\infty} |Q_i - Q_{i-1}| \,.
$$

This claim is easy to prove when $q(x)$ is continuous. In that case, the mean value theorm of calculus, tells us that, $q(x)$ actually *equals* its mean value Q_i at a point in each cell $\mathcal{C}_i$. Let this point be denoted ξ_i^m (where m stands for "mean"). Then we have that

$$
\begin{aligned}
\text{TV}(Q_i) &= \sum_{i=-\infty}^{\infty} |Q_i - Q_{i-1}| \\
&= \sum_{i=-\infty}^{\infty} |q(\xi_i^m) - q(\xi_{i-1}^m)| \\
&\le \sup_\xi \sum_{j=1}^{N} |q(\xi_j) - q(\xi_{j-1})| \,.
\end{aligned}
$$

Where the last inequality holds because the points ξ_i^m are certainly a *specific* set of partition points which then must be less than the expression computing the supremum over all possible partition points.

Problem 6.5 (reconstructions using the minmod slope are TVD)

We desire to prove that when we calculate a piecewise linear reconstruction $\tilde{q}^n(\cdot, t_n)$, using minmod slopes this process can not increase the total variation, i.e.

$$\text{TV}(\tilde{q}^n(\cdot, t_n)) \leq \text{TV}(Q^n)\,.$$

The slopes σ_i^n are computed using

$$\sigma_i^n = \text{minmod}\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta}, \frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\,,$$

where the minmod function is defined by

$$\text{minmod}(a, b) = \begin{cases} a & \text{if} \quad |a| < |b| \quad \text{and} \quad ab > 0 \\ b & \text{if} \quad |b| < |a| \quad \text{and} \quad ab > 0 \\ 0 & \text{if} \quad ab \leq 0 \end{cases}$$

Our reconstruction $\tilde{q}^n(\cdot, t_n)$ is piecewise linear reconstruction i.e. in cell $\mathcal{C}_i$ it is defined by

$$\tilde{q}^n(\cdot, t_n) = Q_i^n + \sigma_i^n(x - x_i)\,.$$

With these definitions we can now compute the total variation $\text{TV}(\tilde{q}^n(\cdot, t_n))$ as

$$\begin{aligned} \text{TV}(\tilde{q}^n(\cdot, t_n)) &= \int_{-\infty}^{\infty} |\frac{d\tilde{q}^n}{d\xi}(\xi, t_n)|d\xi \\ &= \sum_{i=-\infty}^{\infty} \int_{C_i} |\frac{d\tilde{q}^n}{d\xi}(\xi, t_n)|d\xi \\ &= \sum_{i=-\infty}^{\infty} |\sigma_i^n|\Delta x \\ &= \Delta x \sum_{-\infty}^{\infty} |\sigma_i^n| \\ &\leq \Delta x \sum_{i=-\infty}^{\infty} \left|\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=-\infty}^{\infty} \left| Q_i^n - Q_{i-1}^n \right| \\
&\equiv \text{TV}(Q^n)\,,
\end{aligned}
$$

and thus proving the claim. In the above we have used the fact that the minmod function is smaller than either of its two arguments and is smaller than its first argument in particular

$$
|\text{minmod}(a, b)| \leq |a|\,.
$$

Problem 6.6 (resulting methods for some special flux limiter functions)

Defining $\delta_{i-1/2}^n = Q_i^n - Q_{i-1}^n$ and with LeVeque Eq. 6.32 we have

$$
F_{i-1/2}^n = \bar{u}^- Q_i^n + \bar{u}^+ Q_{i-1}^n + \frac{1}{2}|\bar{u}| \left(1 - \left|\frac{\Delta t \bar{u}}{\Delta x}\right|\right) (Q_i^n - Q_{i-1}^n)
$$

That this expression results in the Lax-Wendroff method is obvious from LeVeque Eq. 6.9, which gives the Lax-Wendroff flux for a REA algorithm with a piecewise linear reconstruction given by

$$
\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n (x - x_i)\,.
$$

The resulting flux in that equation is the same the flux presented above.

Problem 6.7 (an application of Harten's TVD test)

$\bar{u} < 0$, then LeVeque 6.1 to the flux-limiter method 6.41 we have

$$
Q_i^{n+1} = Q_i^n - \nu(Q_{i+1}^n - Q_i^n) + \frac{1}{2}\nu(1+\nu) \left(\phi(\theta_{i+1/2}^n)(Q_{i+1}^n - Q_i^n) - \phi(\theta_{i-1/2}^n)(Q_i^n - Q_{i-1}^n)\right) \tag{62}
$$

Then equation 6.1 we have

$$
Q_i^{n+1} = Q_i^n - C_{i-1}^n (Q_i^n - Q_{i-1}^n) + D_i^n (Q_{i+1}^n - Q_i^n) \tag{63}
$$

write 6.41 in the form required by Thm 6.1, 6.41 becomes

$$
Q_i^{n+1} = Q_i^n + \left(-\nu + \frac{1}{2}\nu(1+\nu) \left(\phi(\theta_{i+1/2}^n) - \phi(\theta_{i-1/2}^n)\right) \frac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n}\right) (Q_{i+1}^n - Q_i^n) \tag{64}
$$

from which we see that

$$\begin{aligned} C^n_{i-1} &= 0 \\ D^n_i &= -\nu + \frac{1}{2}\nu(1+\nu)\left(\phi(\theta^n_{i+1/2}) - \phi(\theta^n_{i-1/2})\right)\frac{Q^n_i - Q^n_{i-1}}{Q^n_{i+1} - Q^n_i} \end{aligned}$$

Now if $\bar{u} < 0$ then

$$\theta^n_{i-1/2} \equiv \frac{\Delta Q^n_{I-1/2}}{\Delta Q^n_{i-1/2}} = \frac{\Delta Q^n_{i+1-1/2}}{\Delta Q^n_{i-1/2}} = \frac{\Delta Q^n_{i+1/2}}{\Delta Q^n_{i-1/2}} = \frac{Q^n_{i+1} - Q^n_i}{Q^n_i - Q^n_{i-1}}$$

Thus

$$D^n_i = -\nu + \frac{1}{2}\nu(1+\nu)\left(\phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}}\right) \tag{65}$$

so we can check the conditions required for the theorem of Harten

$$\begin{aligned} C^n_{i-1} &\geq 0\forall i \qquad (66) \\ D^n_i &\geq 0\forall i \qquad (67) \\ C^n_i + D^n_i &\leq 1\forall i \Rightarrow D^n_i \leq 1 \qquad (68) \end{aligned}$$

therefore $0 \leq D^n_i \leq 1$ would require that

$$0 \leq -\nu + \frac{1}{2}\nu(1+\nu)\left(\phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}}\right) \leq 1 \tag{69}$$

which implys that

$$\frac{2\nu}{\nu(1+\nu)} \leq \phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}} \leq \frac{2(1+\nu)}{\nu(1+\nu)} \tag{70}$$

equivalently to

$$\frac{2}{(1+\nu)} \leq \phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}} \leq \frac{2}{\nu} \tag{71}$$

If the CFL condition holds then $-1 \leq \nu \leq 0$ then we have bounds on $2/(\nu+1)$ by the following manipulations

$$\begin{aligned} 0 \leq &\quad \nu + 1 &\leq 1 \\ 1 \leq &\quad \tfrac{1}{\nu+1} &\leq \infty \\ 2 \leq &\quad \tfrac{2}{\nu+1} &\leq \infty \end{aligned}$$

and a bound on $2/\nu$ given by

$$\begin{array}{rcl} -1 \geq & \frac{1}{\nu} & \geq -\infty \\ -\infty \leq & \frac{1}{\nu} & \leq -1 \\ -\infty \leq & \frac{2}{\nu} & \leq -2 \end{array}$$

If we require that

$$2 \leq \left| \phi(\theta^n_{i+1/2}) - \frac{\phi(\theta^n_{i-1/2})}{\theta^n_{i-1/2}} \right| \tag{72}$$

then the above will always be satisfied. Thus we should require

$$2 \leq \left| \phi(\theta_1) - \frac{\phi(\theta_2)}{\theta_2} \right| \tag{73}$$

for all θ_1 and θ_2.

Problem 6.8 (symmetry requirements on limiter functions)

Problem 6.9 (high resolution fluxes for systems)

LeVeque Eq. 6.48 is given by

$$\mathcal{F}(Q_{i-1}, Q_i) = \frac{1}{2} A(Q_{i-1} + Q_i) - \frac{1}{2} \frac{\Delta t}{\Delta x} A^2 (Q_i - Q_{i-1})\,.$$

Which we will first convert into the equation LeVeque Eq. 6.49. Since $A = A^+ + A^-$ and $|A| = A^+ - A^-$, consider first the manipulations of

$$A(Q_{i-1} + Q_i)\,.$$

We have

$$\begin{array}{rcl} A(Q_{i-1} + Q_i) & = & (A^+ + A^-)(Q_{i-1} + Q_i) \\ & = & A^+ Q_{i-1} + A^+ Q_i + A^- Q_{i-1} + A^- Q_i \\ & = & A^+ Q_{i-1} + \left(A^+ Q_{i-1} - A^+ Q_{i-1} \right) + A^+ Q_i \\ & + & A^- Q_{i-1} + A^- Q_i + \left(A^- Q_i - A^- Q_i \right) \\ & = & 2A^+ Q_{i-1} + 2A^- Q_i - A^+ (Q_{i-1} - Q_i) + A^- (Q_{i-1} - Q_i) \\ & = & 2A^+ Q_{i-1} + 2A^- Q_i - (A^+ - A^-)(Q_{i-1} - Q_i) \\ & = & 2A^+ Q_{i-1} + 2A^- Q_i + |A| (Q_i - Q_{i-1})\,. \end{array}$$

If this gives the first three terms from LeVeque Eq.6.49 how can any manipulations of A^2 given me the remaining.

Next consider the expression A^2, we have

$$
\begin{aligned}
A^2 &= (A^+ + A^-)^2 \\
&= (A^+ - A^- + A^- + A^-)(A^+ + A^-) \\
&= \left(|A| - \frac{\Delta t}{\Delta x}(|A| + 2A^-)(A^+ + A^-) \right)(Q_i - Q_{i-1})
\end{aligned}
$$

Now we can simplify $A^+ + A^-$ in two different ways

$$
A^+ + A^- = \begin{cases} |A| + A^- + A^- = |A| + 2A^- \\ A^+ + |A| + A^+ = |A| + 2A^+ \end{cases}
$$

$$
\begin{aligned}
\mathcal{F}(Q_{i-1}, Q_i) &= A^+ Q_{i-1} + A^- Q_i + \frac{1}{2}\left(|A| - \frac{\Delta t}{\Delta x}(|A| + 2A^-)(|A| + 2A^+) \right)(Q_i - Q_{i-1}) \\
&= A^+ Q_{i-1} + A^- Q_i + \frac{1}{2}\left(|A| - \frac{\Delta t}{\Delta x} \right)
\end{aligned}
$$

To show that LeVeque Eq. 6.48 gives the Lax-Wendroff method insert this expression into our general flux-differencing method formulation

$$
Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}(Q_i, Q_{i+1}) - \mathcal{F}(Q_{i-1}, Q_i))
$$

we have that

$$
\begin{aligned}
Q_i^{n+1} &= Q_i^n \\
&- \frac{\Delta t}{\Delta x}\left(\frac{1}{2} A (Q_i^n + Q_{i+1}^n - Q_{i-1}^n - Q_i^n) - \frac{1}{2}\frac{\Delta t}{\Delta x} A^2 (Q_{i+1}^n - Q_i^n - Q_i^n + Q_i^n) \right) \\
&= Q_i^n - \frac{\Delta t}{2\Delta x}\left(A(Q_{i+1}^n - Q_{i-1}^n) \right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x} \right)^2 A^2 (Q_{i+1}^n - 2Q_i^n + Q_i^n)) ,
\end{aligned}
$$

which is the Lax-Wendroff method, as we were to show.

Chapter 7 (Boundar Conditions and Ghost Cells)

Problem 7.1 (ghost cell computed for the Lax-Wendroff flux)

The ghost cell specification given by LeVeque Eq. 7.9 is to take

$$Q_0^n = g_0(t_n + \frac{\Delta x}{2\bar{u}})\,.$$

The the Lax-Wendroff flux $F_{i-1/2}$ is given by

$$F_{i-1/2} = (A^- Q_i^n + A^+ Q_{i-1}^n) + \frac{1}{2}|A|(I - \frac{\Delta t}{\Delta x}|A|)(Q_i^n - Q_{i-1}^n)\,,$$

for the pure advection equation we have $A^- = 0$, $A^+ = \bar{u}$, and $|A| = \bar{u}$ so the Lax-Wendroff flux becomes

$$F_{i-1/2} = \bar{u}Q_{i-1}^n + \frac{1}{2}\bar{u}\left(1 - \frac{\Delta t}{\Delta x}\bar{u}\right)(Q_i^n - Q_{i-1}^n)\,.$$

To evaluate $F_{1/2}$ take $i = 1$ in the above giving

$$\begin{aligned}
F_{1/2} &= \bar{u}Q_0^n + \frac{1}{2}\bar{u}\left(1 - \frac{\Delta t}{\Delta x}\right)(Q_1^n - Q_0^n) \\
&= \left(\bar{u} - \frac{1}{2}\bar{u}\left(1 - \frac{\Delta t}{\Delta x}\bar{u}\right)\right)Q_0^n + \frac{1}{2}\bar{u}\left(1 - \frac{\Delta t}{\Delta x}\bar{u}\right)Q_1 \\
&= \frac{\bar{u}}{2}\left(1 + \frac{\Delta t}{\Delta x}\bar{u}\right)Q_0^n + \frac{\bar{u}}{2}\left(1 - \frac{\Delta t}{\Delta x}\bar{u}\right)Q_1 \\
&= \frac{\bar{u}}{2}\left(1 + \frac{\Delta t}{\Delta x}\bar{u}\right)g_0(t_n + \frac{\Delta x}{2\bar{u}}) + \frac{\bar{u}}{2}\left(1 - \frac{\Delta t}{\Delta x}\bar{u}\right)Q_1\,.
\end{aligned}$$

This expression is to be compared with LeVeque Eq. 7.6 which is

$$F_{1/2} = \bar{u}g_0(t_n + \frac{\Delta t}{2})\,.$$

For a CFL number of near one, we have that $\frac{\Delta t}{\Delta x}\bar{u} \approx 1$, equivalently $\frac{\Delta x}{\bar{u}} \approx \Delta t$ and we can see that the Lax-Wendroff flux $F_{1/2}$ derived above is approximatly the same expression as LeVeque Eq. 7.6.

Problem 7.2 (solid-wall ghost cells for acoustics)

Part (a): LeVeque Eq. 7.17 is assigns a state value Q_0 in the first ghost cell relative to the first internal state variable Q_1 given by $p_0 = p_1$ and $u_0 = -u_1$. To solve the Riemann problem introduced at $x_{1/2} = a$ with left state Q_0 and right state Q_1, we begin by decomposing the jump in state $Q_1 - Q_0$ into jumps in characteristic variables α as

$$\begin{aligned} \alpha &= R^{-1}(Q_1 - Q_0) \\ &= \frac{1}{2Z} \begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix} \begin{bmatrix} p_1 - p_0 \\ u_1 - u_0 \end{bmatrix} \\ &= \frac{1}{2Z} \begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix} \begin{bmatrix} 0 \\ 2u_1 \end{bmatrix} \\ &= \frac{1}{2Z} \begin{bmatrix} 2Zu_1 \\ 2Zu_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1 \end{bmatrix} . \end{aligned}$$

Where to avoid confusion with the first ghost cell we have renamed the constant acoustic impledance from its normal Z_0 to Z. So the intermediate state of this Riemann problem q^* has values given by

$$q^* = Q_0 + \alpha^1 r^1 = \begin{bmatrix} p_1 \\ -u_1 \end{bmatrix} + u_1 \begin{bmatrix} -Z \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 - Zu_1 \\ 0 \end{bmatrix} ,$$

showing the value of $u^* = 0$ as expected.

Chapter 11 (Nonlinear Scalar Conservation Laws)

Problem 11.4 (exact solutions to Burgers' equation)

Burgers' equation in conservative form given by

$$u_t + \left(\frac{1}{2} u^2 \right)_x = 0 .$$

Part (a): Our initial conditions are given by

$$u_0(x)$$

Now the characteristics structure of Burger's equation is given by

$$\frac{dx}{dt} = u = \circ u$$

Now we have characteristic crossing immediatly at $x = \pm 1$ and thus we expect shocks to form, at $x = -1$ the speed will be

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{1}{2}(u_l + u_r) = \frac{1}{2}(1 + 0) = \frac{1}{2}$$

at $x = +1$ the speed will be

$$s = -\frac{1}{2}$$

From the characteristic drawing just produced. It looks like that shown in Figure XXX. It looks like the shocks intersect at $x = 0$ and $t = 2$. From this point on we have the following Riemann problem

$$u(x) = \begin{cases} +1 & x < 0 \\ -1 & x > 0 \end{cases}$$

which has a third shock traveling at the speed of

$$s = \frac{1}{2}(1 - 1) = 0$$

The full characteristc structure is the following

Then $u(x,t)$ is given by for $t < 2$ as

$$u(x,t) = \begin{cases} 1 & x < \frac{1}{2}(t-2) \\ 0 & \frac{1}{2}(t-2) < x < -\frac{1}{2}(t-2) \\ -1 & x > -\frac{1}{2}(t-2) \end{cases}$$

and when $t > 2$ our $u(x,t)$ is given by

$$u(x,t) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases}$$

Our initial condtitions are given by

$$u(x) = \begin{cases} -1 & x < -1 \\ 0 & -1 < x < 1 \\ 1 & x > 1 \end{cases}$$

The characteristic structure of this problem is given

$$\begin{aligned} \frac{dx}{dt} &= -1 \quad x < -1 \\ \frac{dx}{dt} &= 0 \quad -1 < x < 1 \\ \frac{dx}{dt} &= 1 \quad x > 1 \end{aligned}$$

wiht rarefaction fans at $x = \pm 1$. The solution to $u_t + uu_x = 0$ in each rarefaction fas is given by

$$f'(\tilde{q}(x/t)) = x/t$$

for Burgers equatio $f'(x) = x$, so the above is given by $\tilde{q}(x/t) = x/t$. Thus

$$\tilde{q}(x/t) = \begin{cases} -1 & x/t < -1 \\ x/t & -1 < x/t < 0 \\ 0 & x/t > 0 \end{cases}$$

this gives

$$\tilde{q}(x/t) = \begin{cases} -1 & x < -t \\ x/t & -t \le x \le 0 \\ 0 & x > 0 \end{cases}$$

For the centered rarefaction at $x = +1$ we have

$$\tilde{q}(x/t) = \begin{cases} 0 & x < 0 \\ x/t & 0 \le x/t \le 1 \\ 1 & x > t \end{cases}$$

Problem 11.5 (interacting shocks and fans in Burgers' equation)

$$u(x) = \begin{cases} 2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The characteristic structure for Buger's equation is given by $\frac{dx}{dt} = u$. Thus a rarefaction ran forms at $x = 0$, and a shock forms at $x = 1$. The Rankine-Hugoniot shock speeds for Burger's equation is given by

$$s = \frac{1}{2}(u_l + u_r) = \frac{1}{2}(2 + 0) = 1\,.$$

The edges of the rarefaction travel at speeds +2, and will therefore intersect the shock at some point. Inside the rareaction fan the solution is given by

$$f'(q(x/t)) = x/t$$

From LeVeque Eq. 11.27 this is valid for scalar conservation laws. For the Burger's equation $f'(q) = q$ so the above becomes

$$q(x/t) = x/t$$

Thus the solution structure looks like. Now the T_c, is given by $x = 2t$ and $x = t + 1$ intersect when $2T_c = T_c + 1$, or $T_c = 1$.

$$u(x,t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < 2t \\ 2 & 2t < x < t+1 \\ 0 & x > t+1 \end{cases}$$

Then when $t > T_c$ we must evaluate $x_s(t)$

Part (a): The Rankine-Hugnoit equations gives

$$\frac{x_s(t)}{dt} = \frac{1}{2}(u_l + u_r) = \frac{1}{2}(\frac{x_s}{t} + 0)$$

which gives

$$\frac{dx_s}{dt} = \frac{1}{2}\frac{x_s}{t}$$

or

$$\begin{aligned} \frac{dx_s}{x_s} &= \frac{dt}{2t} \\ \ln(x_s(t)) + C &= \frac{1}{2}\ln(t) \\ x_s &= C\sqrt{t} \end{aligned}$$

with $x_s(t = 1) = 2 = C$, which gives $C = 2$, so $x_s(t) = 2\sqrt{t}$.

Part (b): Now I expect that the wave p propagated as if there were no iterface between the rarefaction

Chapter 13 (Nonlinear systems of Conservation Laws)

Problem 13.1 (integral and Hugoniot curves for shallow water)

Part (a): The integral curves of the p-th family are solutions to the following system of ODE's

$$\frac{d\tilde{q}(\xi)}{d\xi} = \alpha(\xi) r^p(\tilde{q}(\xi)) \tag{74}$$

Assuming a unit normalization factor $\alpha(\xi)$ and expressing this ODE system in terms of its components as in the text, we arrive at the following expressions for the 1-integral curve for the shallow water equations

$$\begin{aligned} \tilde{q}^1(\xi) &= \xi \\ \tilde{q}^2(\xi) &= \xi u_* + 2\xi(\sqrt{gh_*} - \sqrt{g\xi}) \end{aligned} \tag{75}$$

These are effectively LeVeque Eq. 13.28 and 13.30 respectively. The slope tangent to these integral curves in the q^1-q^2 plane is equal to

$$\frac{dq^2}{dq^1} = \frac{dq^2/d\xi}{dq^1/d\xi} = \frac{u_* + 2(\sqrt{gh_*} - \sqrt{g\xi}) + 2\xi(\frac{-1}{2}\sqrt{g}\xi^{-1/2})}{1} \tag{76}$$

Simplifying some this gives

$$\frac{dq^2}{dq^1} = \frac{\xi}{\xi}(u_* + 2\sqrt{gh_*} - \sqrt{g\xi}) - \sqrt{g\xi} = \frac{\tilde{q}^2}{\tilde{q}^1} - \sqrt{g\tilde{q}^1} \tag{77}$$

Remembering the definitions of the conservative variables in the shallow water equations we have $q^1 = h$ and $q^2 = hu$ and the above becomes

$$\frac{dq^2}{dq^1} = u - \sqrt{gh} = \lambda^1 \tag{78}$$

Part (b): Consider two points on the 1-shock curve for the shallow water equations, say q^l and q^r. We will assume that q^r is connected from $q^l = q_*$ by some parameter α. Following the discussion on shock waves and the Hugoniot loci for the shallow water equations we obtain for the 1-shocks

$$h^r = h_* + \alpha \tag{79}$$

$$h^r u^r = h_* u_* + \alpha \left[u_* - \sqrt{gh_*(1 + \frac{\alpha}{h_*})(1 + \frac{\alpha}{2h_*})} \right] \tag{80}$$

Then the slope between the two points q^r and q^l is given by

$$\text{slope} = \frac{h_* u_* + \alpha \left[u_* - \sqrt{gh_*(1 + \frac{\alpha}{h_*})(1 + \frac{\alpha}{2h_*})} \right] - h_* u_*}{h_* + \alpha - h_*} \tag{81}$$

simplifying some we get

$$\text{slope} = u_* - \sqrt{gh_*(1 + \frac{\alpha}{h_*})(1 + \frac{\alpha}{2h_*})} \tag{82}$$

Replacing α with $\alpha = h - h_*$ and simplifying some we obtain

$$\text{slope} = u_* - h\sqrt{\frac{g}{2}(\frac{1}{h} + \frac{1}{h_*})} \tag{83}$$

Now to show that this is equivalent to the shock speed between the two states we consider the expression derived relating the velocity behind a shock as a function of the depth h behind the shock (Eq. 13.17 from the book). The 1-shocks must satisfy

$$u(h) = u_* - \sqrt{\frac{g}{2}(\frac{(h_*)^2 - h^2}{hh_*})(h_* - h)} \tag{84}$$

Factoring out $h_* - h$ we obtain for $u(h)$

$$u(h) = u_* - \sqrt{\frac{g}{2}(\frac{1}{h} + \frac{1}{h_*})}|h_* - h| \tag{85}$$

With this expression we can write the shock speed as

$$s = \frac{h_* u_* - hu}{h_* - h} \tag{86}$$

and using the expression above for $u = u(h)$ and remembering that $h = h_* + \alpha$ on the back side of a 1-shock we obtain for s

$$s = \frac{h_* u_* - (h_* + \alpha)(u_* - \sqrt{\frac{g}{2}(\frac{1}{h} + \frac{1}{h_*})}|h_* - h|)}{h_* - h} \tag{87}$$

after simplification

$$s = u_* - h\sqrt{\frac{g}{2}(\frac{1}{h} + \frac{1}{h_*})} \tag{88}$$

Since this is the same expression as we obtained for the slope above we have the desired equivalence.

Problem 13.2 (conservative v.s. primative wave curves)

Figure 13.15 in the text shows integral curves through the points q_r and q_l plotted in the $(h, hu) = (q^1, q^2)$ plane. In particular they are $q_r = (1.0, 0.5)$ and $q_l = (1.0, -0.5)$. To construct the integral wave curves, through the state q_l we draw the 1-integral wave curves and through the state q_r we draw the 2-integral wave curves. For the 1-integral wave curves we have

$$u = u_* + 2(\sqrt{gh_*} - \sqrt{gh}) \tag{89}$$

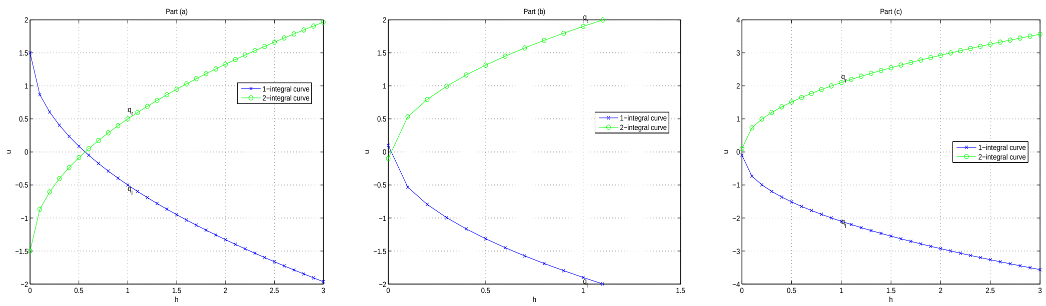

Figure 3: One integral (through q^l) and two integral wave curves (through q^r) for the three parts of Problem 2. The portion of the wave curves where $h > h_l$ or $h > h_r$ is unphysical.

while for the 2-integral wave curves we have

$$u = u_* - 2(\sqrt{gh_*} - \sqrt{gh}) \tag{90}$$

Part (a): For the 1-integral wave curves through $q_l = (1.0, -0.5)$ the above simplifies to (with $g = 1$)

$$u = -0.5 + 2(1.0 - \sqrt{h}) \tag{91}$$

similarly the 2-integral wave curves through $q_r = (1.0, +0.5)$ the above simplifies to (with $g = 1$)

$$u = +0.5 - 2(1.0 - \sqrt{h}) \tag{92}$$

These two integral curves are plotted in the Matlab script `prob_13_2.m` and the output is shown in Figure 3.
Part (b): With $h_l = h_r = 1.0$ and $-u_l = u_r = 1.9$ the integral curves are plotted in `prob_13_2.m`.
Part (c): With $h_l = h_r = 1.0$ and $-u_l = u_r = 2.1$ the integral curves are plotted in `prob_13_2.m`.

Note that as we increase the magnitude of the velocity separation velocity (between the left and right state) the 1 and 2 integral wave curve intersection point becomes closer and closer to 0. For velocities beyond 2.1 the material is moving so fast apart that a "vacuum" state is created between the two.

Problem 13.3 (two rarefactions in shallow water)

The expression for a 2-integral curve is derived in the notes accompanying Page 271 is

$$u = u_* - 2(\sqrt{gh_*} - \sqrt{gh}) \tag{93}$$

For the shallow water equations $\lambda^2 = u + \sqrt{gh} = \frac{q^2}{q^1} + \sqrt{gq^1}$, so the gradient of this expression in the conservative variables $(q^1, q^2) = (h, hu)$ is

$$\nabla \lambda^2 = \begin{bmatrix} -\frac{q^2}{(q^1)^2} + \frac{1}{2}\sqrt{g}(q^1)^{-1/2} \\ \frac{1}{q^1} \end{bmatrix}, \tag{94}$$

this coupled with the 2-wave eigenvector (from Eq. 13.10 in the book) of

$$r^2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{q^2}{q^1} + \sqrt{gq^1} \end{bmatrix} \tag{95}$$

Allows for the calculation of $\nabla \lambda^2 \cdot r^2$. This calculation (after some simplification) produces

$$\nabla \lambda^2 \cdot r^2 = \frac{3}{2}\sqrt{\frac{g}{h}} \neq 0 \tag{96}$$

Since this expression is not equal to zero the 2nd characteristic field of the shallow water equations is generally nonlinear.

Problem 13.4 (1-shock 2-shock collisions)

When a 1-shock collides with at 2-shock in the shallow water equations the Riemann problem that results will produce two new shocks. The left diagram in Figure 4 shows an x-t schematic of this situation and the right diagram in Figure 4 shows this in the (h, hu) plane. One way for this situation to occur is to have the middle state q^m (or the state ahead of the two-shock) connected to the left state by a *two* shock and thus by the discussion in LeVeque (Section 13.7: Shock Waves and Hugoniot Loci) q^l is obtained from q^m (for some $\alpha^l > 0$)

$$q^l = q^m + \alpha^l \begin{bmatrix} 1 \\ u^m - \sqrt{gh^m(1 + \frac{\alpha^l}{h^m})(1 + \frac{\alpha^l}{2h^m})} \end{bmatrix}. \tag{97}$$

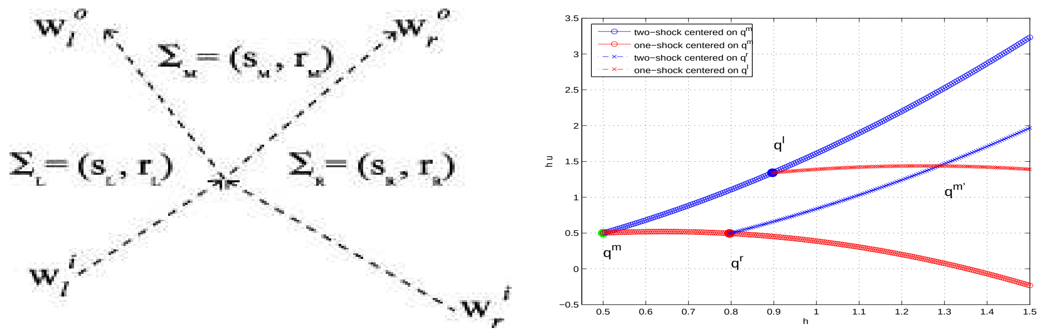

Figure 4: The image to the left is a schematic of a right going or two-shock W_l^i colliding with a left going or one-shock W_r^i and producing two additional shocks, a left going one-shock W_l^o and a right-going two-shock W_r^o. The plot to the right in the above figure shows an example of this behavior in the (h, hu) plane. In this plot the one-shock wave curves are drawn in red while the two-shock wave curves are drawn in blue. This plot is produced using the Matlab script `prob_13_4.m`. See the text for further details.

In the same way, q^m is the state ahead of a one-shock with q^r the state behind. As such it is related to q^r by (for some $\alpha^r > 0$) the following equation

$$q^m = q^m + \alpha^r \left[\frac{1}{u^m + \sqrt{gh^m(1 + \frac{\alpha^r}{h^m})(1 + \frac{\alpha^r}{2h^m})}} \right] . \qquad (98)$$

To provide a numerical example of when this can happen in the (h, hu) plane consider the middle state given by $q^m = (1, 1)$. Now using equation 97 (plotted as $\alpha^l \to \infty$) draw all the possible states behind q^m connected by a two-shock. Since this curve represents *all* states q^l select one (denoted by a blue circle in the plot above). In the same way using equation 98 (plotted as $\alpha^r \to \infty$) draw all the possible states behind q^m connected by a one-shock. Again select a state to be q^r. Through the state q^l draw the loci of one-shocks centered on q^l. Through the state q^r draw the loci of two-shocks centered on q^r. The intersection of these two curves provide the state that is between the two outgoing shocks (denoted $q^{m'}$ in the figure).

WWX: Do I need this description???

Since the first component must be equal we obtain

$$q^l - q^r = \alpha^r - \alpha^l \qquad (99)$$

Equivalently of th second component gives

$$q^l - q^r = \alpha^r \left(u^r + \sqrt{gh^r(1+\frac{\alpha^r}{h^r})(1+\frac{\alpha^r}{2h^r})} \right) - \alpha^l \left(u^l - \sqrt{gh^l(1+\frac{\alpha^l}{h^l})(1+\frac{\alpha^l}{2h^l})} \right) \tag{100}$$

Problem 13.5 (the collision of two rarefactions)

It is *not* possible for two 2-rarefaction to collide with each other. As simple way to see this is as follows. Assume without loss of generality we are considering two left-facing rarefaction fans. Initially, the two rarefaction fans will be separated by a constant state q_m. Since the tail speed of the left most rarefaction fan and the speed of the head of the right most rarefaction fan travel at the *same* speed it is impossible for them to collide. For instance since they both are left-facing fans then the head and tail discussed above *both* travel at the speed given by $\lambda^1(q_m) = u_m - c_m$.

Problem 13.6 (total energy as an entropy function)

Given the definition of the entropy function $\eta(q)$ and the entropy flux $\psi(q)$ for the shallow water equations of

$$\eta(q) = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 \tag{101}$$

$$\psi(q) = \frac{1}{2}hu^3 + gh^2u \tag{102}$$

Part (a): Now $\eta(q)$ is convex if the Hessian matrix $\eta''(q)$ has all positive eigenvalues. We compute the Hessian of η with conservative state of $q = (h, hu) = (q^1, q^2)$. In the (q^1, q^2) coordinates we have η as

$$\eta(q) = \frac{1}{2}\frac{(q^2)^2}{q^1} + \frac{1}{2}g(q^1)^2 \tag{103}$$

Now we first compute the gradient of $\eta(q)$ obtaining

$$\eta'(q) = \nabla_q \eta = \left(-\frac{1}{2}\frac{(q^2)^2}{(q^1)^2} + gq^1, \frac{q^2}{q^1} \right) \tag{104}$$

Then the Hessian is given by

$$\eta''(q) = \begin{bmatrix} \frac{\partial}{\partial q^1}\left(-\frac{1}{2}\frac{(q^2)^2}{(q^1)^2} + gq^1\right) & \frac{\partial}{\partial q^2}\left(-\frac{1}{2}\frac{(q^2)^2}{(q^1)^2} + gq^1\right) \\ \frac{\partial}{\partial q^1}\left(\frac{q^2}{q^1}\right) & \frac{\partial}{\partial q^2}\left(\frac{q^2}{q^1}\right) \end{bmatrix} \tag{105}$$

or simplifying some

$$\begin{bmatrix} \frac{(q^2)^2}{(q^1)^3} + g & -\frac{1}{2}\frac{q^2}{(q^1)^2} \\ -\frac{q^2}{(q^1)^2} + g & \frac{1}{q^1} \end{bmatrix} \tag{106}$$

Now we note that this is symmetric as required. To show that $\eta''(q)$ is positive definite if its eigenvalues are strictly positive. To find its eigenvalues we must evaluate the following

$$\begin{vmatrix} \frac{(q^2)^2}{(q^1)^3} + g - \lambda & -\frac{q^2}{(q^1)^2} \\ -\frac{q^2}{(q^1)^2} + g & \frac{1}{q^1} - \lambda \end{vmatrix} \tag{107}$$

or

$$\left(\frac{(q^2)^2}{(q^1)^3} + g - \lambda\right)\left(\frac{1}{q^1} - \lambda\right) - \frac{q^2}{(q^1)^2}\frac{q^2}{(q^1)^2} = 0 \tag{108}$$

or expanding the terms in the above we get

$$\frac{(q^2)^2}{(q^1)^4} - \lambda\frac{(q^2)^2}{(q^1)^3} + \frac{g}{q^1} - g\lambda - \lambda\frac{1}{q^1} + \lambda^2 - \frac{(q^2)^2}{(q^1)^4} = 0 \tag{109}$$

or

$$\lambda^2 - \lambda\left[\frac{(q^2)^2}{(q^1)^3} + g + \frac{1}{q^1}\right] + \frac{g}{q^1} = 0 \tag{110}$$

or

$$q^1\lambda^2 - \lambda\left[\frac{(q^2)^2}{(q^1)^2} + gq^1 + 1\right] + g = 0 \tag{111}$$

We could work out the general expression for λ but since we are only interested in showing that $\lambda > 0$ lets define a, b, and c as

$$a = q^1 \tag{112}$$

$$b = \left[\left(\frac{q^2}{q^1}\right)^2 + gq^1 + 1\right] \tag{113}$$

$$c = g \tag{114}$$

Our quadratic equation for λ becomes

$$a\lambda^2 - b\lambda + c = 0 \tag{115}$$

Which has its roots given by

$$\lambda = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \tag{116}$$

Now since all of a, b, and c are positive and since $\sqrt{b^2 - 4ac} < b$ we have that $\lambda > 0$ as we desired to show.

Part (b): We desire to show that

$$\eta(q)_t + \psi(q)_x = 0\,. \tag{117}$$

The simplest way to verify this or not is to expand the left hand side of the above equation and see if it can be equated to zero. Specifically, select a set of variables to work in (be they primitive, conservative, or something else) compute the time derivatives using the shallow water equations (with no bottom topography) and substitute into the above expression effectively eliminated all time derivatives in terms of spatial derivatives. If the above expression is true then everything will simplify to a zero. The shallow water equations in primitive variables are

$$h_t + (hu)_x = 0 \tag{118}$$

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0 \tag{119}$$

in terms of time derivatives of the primitive variables these can be simplified to

$$h_t = -uh_x - hu_x \tag{120}$$

$$u_t = -uu_x - gh_x \tag{121}$$

These will be substituted into Eq. 117, when expressed in the primitive variables h and u. Doing so we write the left hand side of Eq. 117 as

$$\left(\frac{1}{2}hu^2 + \frac{1}{2}gh^2\right)_t + \left(\frac{1}{2}hu^3 + gh^2u\right)_x \tag{122}$$

Expanding the derivatives in terms of the product rule we obtain

$$\frac{1}{2}u^2h_t + huu_t + ghh_t + \frac{1}{2}u^3h_x + \frac{3}{2}hu^2u_x + 2ghuh_x + gh^2u_x\,. \tag{123}$$

Replacing the above time derivatives with spatial derivatives compute above we obtain

$$\tfrac{1}{2}u^2\left(-uh_x - hu_x\right) + hu\left(-uu_x - gh_x\right) + gh\left(-uh_x - hu_x\right) + \tag{124}$$

$$\tfrac{1}{2}u^3 h_x + \tfrac{3}{2}hu^2 u_x + 2ghuh_x + gh^2 u_x \,. \tag{125}$$

Which simplifies to zero as expected. To show that $\psi'(q) = \eta'(q)f'(q)$ we consider each expression in tern. Now we have computed $\eta'(q)$ in Eq. 104. In a similar manner in the (q^1, q^2) coordinates we have ψ as

$$\psi(q) = \frac{1}{2}\frac{(q^2)^3}{(q^1)^2} + \frac{1}{2}gq^1q^2 \,. \tag{126}$$

Thus the expression $\psi'(q)$ is given by

$$\psi'(q) = \nabla_q\psi = \left(-\frac{(q^2)^3}{(q^1)^2} + gq^2, \frac{3}{2}\frac{q^2}{q^1} + gq^1\right) \tag{127}$$

$$= \left(-u^3 + ghu, \frac{3}{2}u^2 + gh\right) \tag{128}$$

It remains to calculate $f'(q)$. For the shallow water equations $f(u)$ is given by

$$f(q) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix} \tag{129}$$

Converting this to conservative variables

$$q = \begin{pmatrix} h \\ hu \end{pmatrix} = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}$$

the flux becomes

$$f(q) = \begin{pmatrix} q^2 \\ \frac{(q^2)^2}{q^1} + \frac{1}{2}g(q^1)^2 \end{pmatrix} \tag{130}$$

so $\frac{\partial f}{\partial q}$ is then given by

$$\frac{\partial f}{\partial q} = \begin{pmatrix} 0 & 1 \\ -\frac{(q^2)^2}{(q^1)^2} + gq^1 & 2\frac{q^2}{q^1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix} \,. \tag{131}$$

The entropy flux for the shallow water equations is given by

$$\eta(q) = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 = \frac{1}{2}\frac{(q^2)^2}{q^1} + \frac{1}{2}g(q^1)^2 \tag{132}$$

so the gradient of the entropy flux is given by

$$\eta'(q) = \left(-\frac{1}{2}\left(\frac{q^2}{q^1}\right)^2 + gq^1, \frac{q^2}{q^1}\right) \tag{133}$$

$$= \left(-\frac{1}{2}u^2 + gh, u\right). \tag{134}$$

Now we desire to show that $\psi'(q) = \eta'(q)f'(q)$. To show this we'll compute

$$\eta'(q)f'(q) = \left(-\frac{1}{2}u^2 + gh, u\right)\begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix} \tag{135}$$

$$= \left(-u^3 + ghu, -\frac{1}{2}u^2 + gh + 2u^2\right) \tag{136}$$

$$= \left(-u^3 + ghu, gh + \frac{3}{2}u^2\right). \tag{137}$$

Since this last expression equals $\psi'(q)$ we have shown what was desired.

Problem 13.7 (the p-system)

The p-system given in the text is

$$v_t - u_x = 0 \tag{138}$$

$$u_t + p(v)_x = 0 \tag{139}$$

or in matrix notation we have

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(v) \end{pmatrix}_x = 0\,. \tag{140}$$

Part (a): To derive the characteristic speeds of this system we must first write it in non conservative form i.e.

$$v_t - u_x = 0 \tag{141}$$

$$u_t + p'(v)v_x = 0\,, \tag{142}$$

or again in matrix notation

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix}_x = 0\,. \tag{143}$$

so the Jacobian of the flux function for the p-system is given by

$$f'(q) = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix} . \tag{144}$$

This Jacobian has eigenvalues given by the solution to

$$\begin{vmatrix} -\lambda & -1 \\ p'(v) & -\lambda \end{vmatrix} = 0 \tag{145}$$

or $\lambda^2 + p'(v) = 0$ which has $\lambda^{1,2} = \mp\sqrt{-p'(v)}$ as it solution (using the usual ordering that $\lambda^1 < \lambda^2$). From this we see that the eigenvalues are are real if and only if $p'(v)$ is negative.

Part (b): The Rankine-Hugoniot equation for the p-system between state q and q_* is given by

$$s(q_* - q) = f(q_*) - f(q) \tag{146}$$

in terms of the conservative variables v and u we have

$$s \left[\begin{pmatrix} v_* \\ u_* \end{pmatrix} - \begin{pmatrix} v \\ u \end{pmatrix} \right] = \left[\begin{pmatrix} -u_* \\ p(v_*) \end{pmatrix} - \begin{pmatrix} -u \\ p(v) \end{pmatrix} \right] . \tag{147}$$

Which in component form give the following equations

$$s(v_* - v) = -u_* + u \tag{148}$$

$$s(u_* - u) = p(v_*) - p(v) . \tag{149}$$

Solving for the s in the first equation we have that

$$s = \frac{-u_* + u}{v_* - v} \tag{150}$$

and putting it into the second equation we obtain

$$\frac{-u_* + u}{v_* - v}(u_* - u) = p(v_*) - p(v) . \tag{151}$$

Solving for u in terms of v in the above we perform the following manipulations

$$-\frac{(u_* - u)^2}{(v_* - v)} = p_* - p \tag{152}$$

$$(u_* - u)^2 = (p - p_*)(v_* - v) \tag{153}$$

Which gives for u the following

$$
\begin{aligned}
u &= u_* \pm \sqrt{(p-p_*)(v_*-v)} && (154)\\
&= u_* \pm \sqrt{-\left(\frac{p-p_*}{v-v_*}\right)(v-v_*)^2} && (155)\\
&= u_* \pm \sqrt{-\left(\frac{p-p_*}{v-v_*}\right)}|v-v_*| && (156)
\end{aligned}
$$

since the volume behind a shock must be greater then the state ahead $v > v_*$ and the absolute value can be dropped and we have the requested result from the book.

Part (c): From the first equation we have

$$s(v_* - v) = -u_* + u \quad (157)$$

and putting this in the recently found solution for u gives

$$s(v_* - v) = -u_* + u_* \pm \sqrt{-\frac{p(v) - p(v_*)}{v - v_*}}(v - v_*) \quad (158)$$

which gives

$$s = \mp\sqrt{-\frac{p(v) - p(v_*)}{v - v_*}} \quad (159)$$

Note that for $v \approx v_*$ the difference quotient above is an approximation to the derivative, so $s \approx \lambda$ and the approximation becomes better as $v \to v_*$ from above. Thus we see that the *minus* or top sign corresponds to the 1-wave and the plus or bottom sign corresponds to the 2-waves.

Part (d): Let $q^* = (1, 1)$ for $-3 \leq v \leq 5$ In figure 5 we have plotted the Hugoniots for each of the suggested pressure volume relationships.

Part (e): To determine the *two-shock* solution for the p-system with $p(v) = -e^v$, $q_l = (1, 1)$, and $q_r = (3, 4)$ we can plot the Hugoniot loci given by

$$u = u_* \pm \sqrt{-\left(\frac{p(v) - p(v_*)}{v - v_*}\right)}(v - v_*) \quad (160)$$

for initial states $(v_*, u_*) = (1, 1)$ and $(v_*, u_*) = (3, 4)$ while ensuring to use the correct sign depending on if we are connecting the left or right state to the middle one.

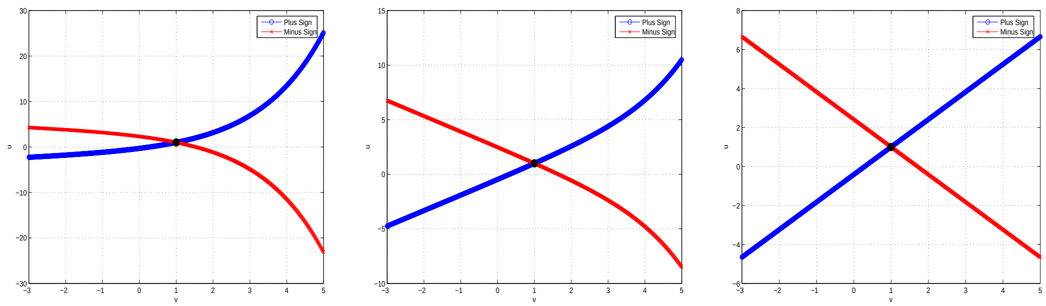

Figure 5: One and Two wave Hugoniot loci for the shallow water equations with equations of state consisting of **(i)** $p(v) = -e^v$, **(ii)** $p(v) = -(2v+0.1e^v)$, and **(iii)** $p(v) = -2v$. The red curve corresponds to the one wave and the blue curve corresponds to the two wave.

We plot the *left* Hugoniot from the state $(v_*, u_*) = (1, 1)$. This is given by taking the *minus* sign in the above equation and we obtain

$$u = u_* - \sqrt{-\left(\frac{p(v) - p(v_*)}{v - v_*}\right)}(v - v_*) \tag{161}$$

or

$$u = 1 - \sqrt{-\left(\frac{-e^v + e^1}{v - 1}\right)}(v - 1) \tag{162}$$

Next we plot the *right* Hugoniot from the state $(v_*, u_*) = (3, 4)$. This is given by taking the *plus* sign in the above equation giving

$$u = u_* + \sqrt{-\left(\frac{p(v) - p(v_*)}{v - v_*}\right)}(v - v_*) \tag{163}$$

or

$$u = 4 + \sqrt{-\left(\frac{-e^v + e^3}{v - 3}\right)}(v - 3) \tag{164}$$

Both of these expressions are plotted in the function `prob_13_7_e.m` and the result is displayed in figure 6 with an estimate of the intersection point marked with a diamond.

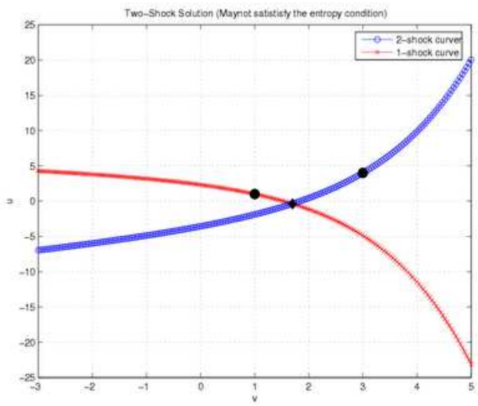

Figure 6: The one Hugoniot loci through the state $(1, 1)$ and the two Hugoniot loci through the state $(3, 4)$ for the shallow water equations with equations of state consisting of $p(v) = -e^v$

(ii): From the the discussion above (v_m, u_m) must be given by the solution to the following system of equations

$$u_m = 1 - \sqrt{-\left(\frac{-e^{v_m} + e}{v_m - 1}\right)(v_m - 1)} \qquad (165)$$

$$u_m = 4 + \sqrt{-\left(\frac{-e^{v_m} + e^3}{v_m - 3}\right)(v_m - 3)} \qquad (166)$$

which by equating the equations for u_m gives a single equation for v_m of

$$1 - \sqrt{-\left(\frac{-e^{v_m} + e}{v_m - 1}\right)(v_m - 1)} = 4 + \sqrt{-\left(\frac{-e^{v_m} + e^3}{v_m - 3}\right)(v_m - 3)} \qquad (167)$$

To solve this equation we can use a newton like method. In `prob_13_7_e_fn.m` we have a Matlab function which is used in the Matlab script `prob_13_7_e.m` to with `fzero` to solve for the explicit root. When this is done the value obtained is given by $(1.693712, -0.373986)$.

Part (f): The Lax-entropy condition states that a discontinuity separating q_l and q_r propagating at speed s satisfying the Lax-entropy conditions if there is an index p such that

$$\lambda^p(q_l) > s > \lambda^p(q_r)\,, \qquad (168)$$

(p-characteristic are impinging) on the discontinuity, while the other characteristics are *crossing* the discontinuity

$$\lambda^j(q_l) < s \quad \text{and} \quad \lambda^j(q_r) < s \quad j < p \tag{169}$$

$$\lambda^j(q_l) > s \quad \text{and} \quad \lambda^j(q_r) > s \quad j > p\,. \tag{170}$$

In other words the shock *splits* the characteristics. For our problem with only two characteristics speeds. The 1-wave must then satisfy

$$\lambda^1(q_l) > s > \lambda^1(q_r) \tag{171}$$

$$\lambda^2(q_l) > s \quad \text{and} \quad \lambda^2(q_r) > s \tag{172}$$

Lets check this condition for the solution found above. In the shallow water equations (and in our case with $p(v) = -e^v$) we have

$$\lambda^{(1,2)} = u \pm \sqrt{-p'(v)} = u \pm e^{v/2} \tag{173}$$

Part (g): For the given left state $q_l = (1, 1)$ for the one shock to satisfy the Lax-entropy conditions we must have

$$\lambda^1(q_l) \quad > \quad \lambda^1(q_m) \tag{174}$$

$$-\sqrt{-p'(v_l)} \quad > \quad -\sqrt{-p'(v_m)} \tag{175}$$

$$\sqrt{-p'(v_l)} \quad < \quad \sqrt{-p'(v_m)} \tag{176}$$

$$\lambda^2(q_l) > s \tag{177}$$

$$\lambda^2(q_m) > s \tag{178}$$

From Eq. 174 for a 1-shock must have

$$-p'(v_l) < -p'(v_m)$$

or

$$p'(v_l) > p'(v_m) \tag{179}$$

with $p(v) = -e^v$ and therefor $p'(v) = -e^v$ the above becomes

$$-e^{v_l} > -e^{v_m}$$

which gives

$$v_l < v_m \tag{180}$$

as the entropy requirement for 1-shocks. For 2-shocks the Lax-entropy condition requires that

$$\lambda^2(q_m) > s > \lambda^2(q_r) \tag{181}$$

and

$$\lambda^1(q_m) < s \quad \text{and} \quad \lambda^1(q_r) < s \tag{182}$$

with our equation of state equation 181 becomes

$$\begin{aligned} \sqrt{-p'(v_m)} &> \sqrt{-p'(v_r)} \\ -p'(v_m) &> -p'(v_r) \\ e^{v_m} &> e^{v_r} \\ v_m &> v_r \end{aligned} \tag{183}$$

Problem 13.8 (the exponential EOS p-system)

The p-system we are considering for this problem is given by

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= 0 \end{aligned} \tag{184}$$

with $p(v) = -e^v$.

Part (a): The procedure described in section 13.8.1 looks for the integral curves of Equation 184, the definition of which is that the p-th integral curves are the solutions to the following ordinary differential equations

$$\tilde{q}'(\xi) = \alpha(\xi) r^p(\tilde{q}(\xi)) \,.$$

For the p-system with arbitrary equation of state the eigenvalues are given by $\lambda^{1,2} = \mp\sqrt{-p'(v)}$, which for this equation of state (since $p'(v) = -e^v$) becomes

$$\lambda^{1,2} = \mp e^{v/2} \,. \tag{185}$$

The eigenvectors for the p-system are given by the non-zero vector solutions r^1 and r^2 to

$$\left[\begin{array}{cc} -\lambda^{1,2} & -1 \\ p'(v) & -\lambda^{1,2} \end{array} \right] \left[\begin{array}{c} r_1^{1,2} \\ r_2^{1,2} \end{array} \right] = 0 \,.$$

or

$$\left[\begin{array}{cc} \pm\sqrt{-p'(v)} & -1 \\ p'(v) & \pm\sqrt{-p'(v)} \end{array} \right] \left[\begin{array}{c} r_1^{1,2} \\ r_2^{1,2} \end{array} \right] = 0$$

For r^1 we have (selecting the + sign)

$$\begin{bmatrix} \sqrt{-p'(v)} & -1 \\ p'(v) & \sqrt{-p'(v)} \end{bmatrix} \begin{bmatrix} r_1^1 \\ r_2^1 \end{bmatrix} = 0\,.$$

Now dividing the second row by $-\sqrt{-p'(v)}$ we obtain

$$\begin{bmatrix} \sqrt{-p'(v)} & -1 \\ \sqrt{-p'(v)} & -1 \end{bmatrix} \begin{bmatrix} r_1^1 \\ r_2^1 \end{bmatrix} = 0\,.$$

Showing the underdetermined nature of this system and giving a single constraint on the vector r^1 of

$$\sqrt{-p'(v)}r_1^1 - r_2^1 = 0$$

or

$$r_2^1 = \sqrt{-p'(v)}r_1^1$$

so the first eigenvector is given by

$$r^1 = \begin{bmatrix} 1 \\ \sqrt{-p'(v)} \end{bmatrix}\,. \tag{186}$$

In a similar way, the 2nd eigenvector is given by

$$r^2 = \begin{bmatrix} 1 \\ -\sqrt{-p'(v)}\,. \end{bmatrix} \tag{187}$$

Which for this equation of state gives

$$r^1 = \begin{bmatrix} 1 \\ e^{v/2} \end{bmatrix} \quad \text{and} \quad r^2 = \begin{bmatrix} 1 \\ -e^{v/2} \end{bmatrix}\,. \tag{188}$$

Back to the problem at hand, the 1-integral curves are solutions to

$$\tilde{q}'(\xi) = \alpha(\xi) r^1(\tilde{q}(\xi))$$

taking $\alpha(\xi) = 1$, for simplicity (other values of α only scale the solution) and specifying to the eigenvectors of the p-system found above we have

$$\tilde{q}'(\xi) = \begin{bmatrix} 1 \\ \sqrt{-p'(v)} \end{bmatrix} \tag{189}$$

with $q'(\xi)$ defined as

$$\tilde{q}'(\xi) = \begin{bmatrix} \frac{dv}{d\xi} \\ \frac{du}{d\xi} \end{bmatrix} \tag{190}$$

the system of ordinary differential equation above becomes

$$\frac{dv}{d\xi} = 1 \tag{191}$$

$$\frac{du}{d\xi} = \sqrt{-p'(v(\xi))}\,. \tag{192}$$

The first equation has a solution given by (ignoring integration constants)

$$v(\xi) = \xi\,,$$

which when put into the second equation above gives

$$\frac{du}{d\xi} = \sqrt{-p'(\xi)}\,. \tag{193}$$

Since in our case $p(v) = -e^v$, the above becomes

$$\frac{du}{d\xi} = e^{\frac{\xi}{2}} \tag{194}$$

which when integrated between two points on the integral curve (ξ_1 and ξ_2) gives

$$u(\xi_2) - u(\xi_1) = 2(e^{\xi_2/2} - e^{\xi_1/2})\,. \tag{195}$$

Defining the point on which we center our rarefaction wave curve as (v_*, u_*) (when $\xi = \xi_1$) we have

$$v(\xi_1) = \xi_1 = v_*$$
$$u(\xi_1) = u_*$$

and the point which changes as we more though the wave curve as (v, u) (when $\xi = \xi_2$) we have

$$v(\xi_2) = \xi_2 = v$$
$$u(\xi_2) = u\,,$$

and Eq. 195 becomes

$$u - u_* = 2(e^{v/2} - e^{v_*/2}) \tag{196}$$

or

$$u = u_* - 2(e^{v_*/2} - e^{v/2}) \tag{197}$$

as was to be shown. From the above equation is is clear that

$$u - 2e^{v/2} = u_* - 2e^{v_*/2}$$

for any two points along a 1-integral wave so defining the function $w^1(q)$ as,

$$w^1(q) \equiv u - 2e^{v/2} \tag{198}$$

we see that $w^1(q)$ is a 1-Riemann invariant for this system.

Part (b): Along a p-th centered rarefaction wave we have that the vector state q must satisfy the following system of ODE's

$$\tilde{q}'(\xi) = \frac{r^p(\tilde{q}(\xi))}{\nabla\lambda^p(\tilde{q}(\xi)) \cdot r^p(\tilde{q}(\xi))} \,. \tag{199}$$

For the p-system with this equation of state we have

$$\begin{aligned} \nabla\lambda^p &= \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u}\right)^T (\mp e^{v/2}) \\ &= \left(\mp\frac{1}{2}e^{v/2}, 0\right)^T \end{aligned} \tag{200}$$

so we have for $\nabla\lambda^p \cdot r^p$ the following expression

$$\begin{aligned} \nabla\lambda^p \cdot r^p &= \left(\mp\frac{1}{2}e^{v/2}, 0\right) \cdot (1, \pm e^{v/2})^T \\ &= \mp\frac{e^{v/2}}{2} \,. \end{aligned} \tag{201}$$

So the **1-centered** rarefaction fan must satisfy

$$\begin{pmatrix} \frac{d\tilde{v}}{d\xi} \\ \frac{d\tilde{u}}{d\xi} \end{pmatrix} = \frac{1}{\frac{-1}{2}e^{\tilde{v}/2}} \begin{pmatrix} 1 \\ e^{\tilde{v}/2} \end{pmatrix} = \begin{pmatrix} -2e^{-\tilde{v}/2} \\ -2 \end{pmatrix} \,. \tag{202}$$

The second equation has solution given by

$$u = -2\xi + A \,, \tag{203}$$

for some constant A. The first equation has a solution given by the following manipulations

$$\begin{aligned} e^{\tilde{v}/2} d\tilde{v} &= -2d\xi \\ 2e^{\tilde{v}/2} &= -2\xi + B \\ e^{\tilde{v}/2} &= -\xi + B \\ \tilde{v} &= 2\log(B - \xi)\,. \end{aligned} \quad (204)$$

Now A and B are determined by the fact that the limits of ξ correspond to the head and tail of the rarefaction fan. For example, we require

$$\text{when} \quad \xi_1 = \lambda^1(q_l) \quad \text{we have} \quad \tilde{q}(\xi_1) = q_l \quad \text{and} \quad (205)$$

$$\text{when} \quad \xi_2 = \lambda^1(q_r) \quad \text{we have} \quad \tilde{q}(\xi_2) = q_r \quad (206)$$

which for this problem becomes

$$\begin{aligned} \xi_1 &= \lambda^1(q_l) = -e^{v_l/2} \quad \tilde{q}(\xi_1) = \begin{pmatrix} \tilde{v}(\xi_1) \\ \tilde{u}(\xi_1) \end{pmatrix} = \begin{pmatrix} 2\log(B - \xi_1) \\ -2\xi_1 + A \end{pmatrix} \\ \xi_2 &= \lambda^1(q_r) = -e^{\tilde{v}_r/2} \quad \tilde{q}(\xi_2) = \begin{pmatrix} \tilde{v}(\xi_2) \\ \tilde{u}(\xi_2) \end{pmatrix} = \begin{pmatrix} 2\log(B - \xi_2) \\ -2\xi_2 + A \end{pmatrix}. \end{aligned}$$

Our "left" boundary condition on the u component is

$$u_l = 2e^{v_l/2} + A$$

giving

$$A = u_l - 2e^{v_l/2}\,. \quad (207)$$

In them same way, the "right" boundary condition has $\xi_2 = -e^{v_r/2}$ and our boundary condition on u is given by

$$u_r = 2e^{v_r/2} + A$$

giving

$$A = u_r - 2e^{v_r/2}\,. \quad (208)$$

Which was to be shown. The same type of manipulations for v gives a value of B from the "left" boundary conditions

$$\begin{aligned} \tilde{v}(\xi_1) &= 2\log(B - \xi_1) \\ v_l &= 2\log(B + e^{v_l/2}) \\ e^{v_l/2} &= B + e^{v_l/2} \\ B &= 0\,. \end{aligned}$$

Applying our "right" boundary condition we have

$$\begin{aligned}\tilde{v}(\xi_2) &= 2\log(B-\xi_2)\\ v_r &= 2\log(B+e^{v_r/2})\\ e^{v_r/2} &= B+e^{v_r/2}\\ B &= 0\end{aligned}$$

so the variation in v through the 1 rarefaction waves are represented by

$$\tilde{v}(\xi) = 2\log(-\xi) \quad \text{for} \quad -e^{v_l/2} \leq \xi \leq -e^{v_r/2} \tag{209}$$

in vector form

$$\tilde{q}(\xi) = \begin{pmatrix} 2\log(-\xi) \\ -2\xi + A \end{pmatrix} \quad \text{for} \quad -e^{v_l/2} \leq \xi \leq -e^{v_r/2} \tag{210}$$

Part (c): To be genuinely nonlinear we must have that

$$\nabla\lambda^p \cdot r^p \neq 0\,. \tag{211}$$

Since for this equation of state $\nabla\lambda^p \cdot r^p = \mp\frac{1}{2}e^{v/2} \neq 0$ both fields are genuinely nonlinear.

Part (d): To determine the the 2-Riemann invariants we consider the 2-integral wave curves which are solutions to the following set of ordinary differential equations.

$$\tilde{q}'(\xi) = \alpha(\xi) r^2(\tilde{q}(\xi))$$

taking $\alpha(\xi) = 1$, for simplicity (other values of α only scale the solution) and specifying to the eigenvectors of the p-system found above we have

$$\tilde{q}'(\xi) = \begin{bmatrix} 1 \\ -e^{v/2} \end{bmatrix} \tag{212}$$

giving the system of ordinary differential equation above becomes

$$\frac{dv}{d\xi} = 1 \tag{213}$$

$$\frac{du}{d\xi} = -e^{v/2}\,. \tag{214}$$

The first equation has a solution given by (ignoring integration constants)

$$v(\xi) = \xi\,,$$

which when put into the second equation above gives

$$\frac{du}{d\xi} = -e^{\xi/2} \,. \tag{215}$$

which when integrated between two points on the integral curve (ξ_1 and ξ_2) gives

$$u(\xi_2) - u(\xi_1) = -2(e^{\xi_2/2} - e^{\xi_1/2}) \,. \tag{216}$$

Defining the point on which we center our rarefaction wave curve as (v_*, u_*) (when $\xi = \xi_1$) we have from Eq. 216

$$u - u_* = -2(e^{v/2} - e^{v_*/2}) \tag{217}$$

or

$$u = u_* + 2(e^{v_*/2} - e^{v/2}) \,,$$

which is a representation of the 2-integral curves in the $u - v$ plane. From the above equation is is clear that

$$u + 2e^{v/2} = u_* + 2e^{v_*/2}$$

for any two points along a 2-integral wave so defining the function $w^2(q)$ as,

$$w^2(q) \equiv u + 2e^{v/2} \tag{218}$$

we see that $w^2(q)$ is a 2-Riemann invariant for this system and is the first part of this question. For the second part to this problem, the 2-centered rarefaction fan must satisfy the following system of ODE's

$$\begin{pmatrix} \frac{d\tilde{v}}{d\xi} \\ \frac{d\tilde{u}}{d\xi} \end{pmatrix} = \frac{1}{\frac{1}{2}e^{\tilde{v}/2}} \begin{pmatrix} 1 \\ -e^{\tilde{v}/2} \end{pmatrix} = \begin{pmatrix} 2e^{-\tilde{v}/2} \\ -2 \end{pmatrix}$$

These equations are the 2-wave generalizations to equation 199. Integrating the second equation gives

$$\tilde{u} = -2\xi + C \,,$$

while integrating the first equation gives

$$\begin{aligned} e^{\tilde{v}/2} d\tilde{v} &= 2d\xi \\ 2e^{\tilde{v}/2} &= 2\xi + D \\ \tilde{v}(\xi) &= 2\log(D + \xi) \end{aligned}$$

Now C and D are determined by the fact that the limits of ξ correspond to the head and tail of the 2nd rarefaction fan. For example, we require

$$\begin{array}{lllll} \text{when} & \xi_1 & = & \lambda^2(q_l) & \text{we have} \quad \tilde{q}(\xi_1) = q_l \quad \text{and} \\ \text{when} & \xi_2 & = & \lambda^2(q_r) & \text{we have} \quad \tilde{q}(\xi_2) = q_r \end{array}$$

which for this problem becomes

$$\begin{aligned} \xi_1 &= \lambda^2(q_l) = +e^{v_l/2} \quad \tilde{q}(\xi_1) = \begin{pmatrix} \tilde{v}(\xi_1) \\ \tilde{u}(\xi_1) \end{pmatrix} = \begin{pmatrix} 2\log(D+\xi_1) \\ -2\xi_1 + C \end{pmatrix} \\ \xi_2 &= \lambda^2(q_r) = +e^{v_r/2} \quad \tilde{q}(\xi_2) = \begin{pmatrix} \tilde{v}(\xi_2) \\ \tilde{u}(\xi_2) \end{pmatrix} = \begin{pmatrix} 2\log(D+\xi_2) \\ -2\xi_2 + C \end{pmatrix}. \end{aligned}$$

Our "left" boundary condition on the u component is

$$u_l = -2e^{v_l/2} + C$$

giving

$$C = u_l + 2e^{v_l/2}. \tag{219}$$

In the same way, the "right" boundary condition has a boundary condition on u is given by

$$u_r = -2e^{v_r/2} + C$$

giving

$$C = u_r + 2e^{v_r/2}. \tag{220}$$

Which was to be shown. The same type of manipulations for v gives a value of D from the "left" boundary conditions

$$\begin{aligned} \tilde{v}(\xi_1) &= 2\log(D+\xi_1) \\ v_l &= 2\log(D+e^{v_l/2}) \\ e^{v_l/2} &= D + e^{v_l/2} \\ D &= 0. \end{aligned}$$

Applying our "right" boundary condition we have

$$\begin{aligned} \tilde{v}(\xi_2) &= 2\log(D-\xi_2) \\ v_r &= 2\log(D+e^{v_r/2}) \\ e^{v_r/2} &= D + e^{v_r/2} \\ D &= 0, \end{aligned}$$

so the variation in v through the 2 rarefaction waves are represented functionally by

$$\tilde{v}(\xi) = 2\log(\xi) \quad \text{for} \quad e^{v_l/2} \leq \xi \leq e^{v_r/2} \tag{221}$$

in vector form

$$\tilde{q}(\xi) = \begin{pmatrix} 2\log(\xi) \\ -2\xi + C \end{pmatrix} \quad \text{for} \quad e^{v_l/2} \leq \xi \leq e^{v_r/2} \tag{222}$$

Where the constant C is given by equation 219 or equation 220.

Part (e): To determine the middle state q_m using only centered rarefaction fans as suggested in the text we must connect the left state q_l to q_m along a 1-rarefaction fan, and q_m to q_r along a 2-rarefaction fan. The first constraint requires (using the 1-Riemann invariant from Eq. 198)

$$u_l - 2e^{v_l/2} = u_m - 2e^{v_m/2} \tag{223}$$

connecting the middle state to the right state using a 2-rarefaction wave in the same way (using the 2-Riemann invariant from Eq. 218) we obtain

$$u_r + 2e^{v_r/2} = u_m + 2e^{v_m/2} \tag{224}$$

Adding these two equations gives a for u_m the expression of

$$u_m = \frac{1}{2}\left(u_l + u_r + 2(e^{v_r/2} - e^{v_l/2})\right) . \tag{225}$$

Subtracting these two equations gives an expression for $e^{v_m/2}$ of

$$e^{v_m/2} = \frac{1}{4}\left(u_r - u_l + 2(e^{v_r/2} - e^{v_l/2})\right) . \tag{226}$$

Which can easily be solved for v_m.

Part (f): For the value of q_m computed in part e to be a valid entropy satisfying solution we must have the that characteristic speeds increase as we move through *each* rarefaction fan. What this means is that we must have

$$\lambda^1(q_l) < \lambda^1(q_m) \tag{227}$$

and

$$\lambda^2(q_m) < \lambda^2(q_r) \tag{228}$$

In terms of the known eigenvalues for this version of the p-system we have the above

$$-e^{v_l/2} < -e^{v_m/2}$$

and

$$e^{v_m/2} < e^{v_r/2}$$

Since we have computed $e^{v_m/2}$ in Eq. 226 we can evaluate each of these expressions above in terms of only q_l and q_r. The first equation is then given by

$$e^{v_l/2} > \frac{1}{4}\left(u_r - u_l + 2(e^{v_r/2} - e^{v_l/2})\right) . \tag{229}$$

which simplifies to

$$u_r - u_l + 2(e^{v_r/2} - e^{v_l/2}) < 0 \tag{230}$$

The second equation is then given by

$$\frac{1}{4}\left(u_r - u_l + 2(e^{v_r/2} - e^{v_l/2})\right) < e^{v_r/2} . \tag{231}$$

which simplifies to

$$u_r - u_l + 2(-e^{v_r/2} + e^{v_l/2}) < 0 \tag{232}$$

Adding Eq. 230 to Eq. 232 gives

$$u_r < u_l \tag{233}$$

and subtracting Eq. 230 to Eq. 232 gives

$$e^{v_r/2} - e^{v_l/2} < 0 \tag{234}$$

or

$$v_r < v_l \tag{235}$$

as another condition. In summary then to have an all rarefaction solution to the Riemann problem for this specific p-system the initial conditions must satisfy

$$v_l > v_r \tag{236}$$

$$u_l > u_r \tag{237}$$

Problem 13.9 (genuine nonlinear requirements for the p-system)

The p-system of Exercise 13.7 has eigenvalues given by

$$\lambda^{1,2} = \mp\sqrt{-p'(v)} \tag{238}$$

with corresponding eigenvalues given by

$$r^1 = \begin{pmatrix} 1 \\ +\sqrt{-p'(v)} \end{pmatrix} \quad \text{and} \quad r^2 = \begin{pmatrix} 1 \\ -\sqrt{-p'(v)} \end{pmatrix}. \tag{239}$$

For the p-th field to be generally nonlinear we must have

$$\nabla\lambda^p \cdot r^p \neq 0 \tag{240}$$

In the case above with

$$q = \begin{pmatrix} v \\ u \end{pmatrix} \tag{241}$$

we have that

$$\begin{aligned} \nabla\lambda^1 &= \left(\frac{\partial\lambda^1}{\partial v}, \frac{\partial\lambda^1}{\partial u}\right) \\ &= \left((-1)\frac{1}{2}(-p'(v))^{-1/2}(-p''(v)), 0\right) \\ &= \left(\frac{p''(v)}{2\sqrt{-p'(v)}}, 0\right) \end{aligned} \tag{242}$$

In a similar way we have that

$$\nabla\lambda^2 = \left(\frac{-p''(v)}{2\sqrt{-p'(v)}}, 0\right) \tag{243}$$

so we then have that

$$\nabla\lambda^p \cdot r^p = \left(\frac{\pm p''(v)}{2\sqrt{-p'(v)}}, 0\right) \cdot \left(1, \pm\sqrt{-p'(v)}\right) \tag{244}$$

$$= \frac{\pm p''(v)}{2\sqrt{-p'(v)}} \neq 0 \tag{245}$$

or equivalently $p''(v) \neq 0$ as the condition that must be satisfied for the 1 and 2 fields to be genuinely nonlinear. It is interesting that we can construct a family of $p = p(v)$ relationships that *will* be linearly degenerate by solving $p''(v) = 0$ or $p(v) = A + Bv$.

Problem 13.10 (genuinely nonlinear requirements for 1D elastics)

LeVeque Eq. 2.97 are given by

$$\begin{aligned} \epsilon_t - u_x &= 0 \\ \rho u_t - \sigma(\epsilon)_x &= 0 \end{aligned} \tag{246}$$

with ρ a constant. The second equation can be written as

$$u_t - \frac{\sigma'(\epsilon)}{\rho}\epsilon_x = 0\,. \tag{247}$$

and for constant ρ our flux function is given by

$$f(q) = \begin{bmatrix} u \\ -\frac{\sigma(\epsilon)}{\rho} \end{bmatrix} \tag{248}$$

Defining primitive variables as

$$q = \begin{bmatrix} \epsilon \\ u \end{bmatrix}\,, \tag{249}$$

and Equation 246 can be written in nonconservative form as

$$\begin{bmatrix} \epsilon \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ -\frac{\sigma'(\epsilon)}{\rho} & 0 \end{bmatrix} \begin{bmatrix} \epsilon \\ u \end{bmatrix}_x = 0\,. \tag{250}$$

The Jacobian is given by

$$A(q) = \begin{bmatrix} 0 & -1 \\ -\frac{\sigma'(\epsilon)}{\rho} & 0 \end{bmatrix} \tag{251}$$

which is the same for the p-system with p' replaced by $\frac{\sigma'}{\rho}$. Thus the eigenvalues are given by

$$\lambda^{1,2} = \mp\sqrt{-\frac{\sigma'(\epsilon)}{\rho}} \tag{252}$$

with corresponding eigenvectors given by

$$r^{1,2} = \begin{bmatrix} 1 \\ \pm\sqrt{-\frac{\sigma'(\epsilon)}{\rho}} \end{bmatrix} \tag{253}$$

to be genuinely nonlinear we must have

$$\nabla \lambda^p \cdot r^p \neq 0\,. \tag{254}$$

For this problem we have that

$$\begin{aligned}
\nabla \lambda^p &= \left(\frac{\partial \lambda^p}{\partial \epsilon}, \frac{\partial \lambda^p}{\partial u} \right)^T & (255)\\
&= \left(\mp \frac{1}{2} \left(\frac{-\sigma'(\epsilon)}{\rho} \right)^{-1/2} \left(-\frac{\sigma''(\epsilon)}{\rho} \right), 0 \right)^T & (256)\\
&= \left(\pm \frac{1}{2\rho} \frac{\sigma''(\epsilon)}{\left(\frac{-\sigma'(\epsilon)}{\rho} \right)^{1/2}}, 0 \right)^T\,, & (257)
\end{aligned}$$

so that

$$\begin{aligned}
\nabla \lambda^p \cdot r^p &= \left(\pm \frac{\sigma''(\epsilon)}{2\rho} \left(\frac{\rho}{-\sigma'(\epsilon)} \right)^{1/2}, 0 \right)^T \cdot \left(1, \pm \sqrt{-\frac{\sigma'(\epsilon)}{\rho}} \right) & (258)\\
&= \pm \frac{\sigma''(\epsilon)}{2\rho} \left(\frac{\rho}{-\sigma'(\epsilon)} \right)^{1/2} & (259)
\end{aligned}$$

Giving as our condition for genuine nonlinearity of

$$\pm \frac{\sigma''(\epsilon)}{2\rho} \left(\frac{\rho}{-\sigma'(\epsilon)} \right)^{1/2} \neq 0 \tag{260}$$

which reduces to that of $\sigma''(\epsilon) \neq 0$ for every ϵ.

From the graph given in Figure 2.3 (a), it appears that $\sigma''(\epsilon) = 0$ at the points where the rate of increase of $\sigma(\epsilon)$ slows down. Thus this system is not genuinely nonlinear in this case.

Problem 13.11 (variable coefficient advection)

Our original conservative spatially varying advection equation, given by

$$q_t + (u(x)q)_x = 0\,,$$

can be written as the following hyperbolic system

$$\begin{aligned} q_t + (u(x)q)_x &= 0 \\ u_t &= 0 \end{aligned} \quad (261)$$

Part (a): Define our systems state v, as

$$v = \begin{bmatrix} q \\ u \end{bmatrix}, \quad (262)$$

our system flux is then given by

$$f(v) = \begin{bmatrix} uq \\ 0 \end{bmatrix}, \quad (263)$$

with a Jacobian given by

$$\frac{\partial f}{\partial v} = \begin{bmatrix} \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial q} & \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} u & q \\ 0 & 0 \end{bmatrix}, \quad (264)$$

where f_1 and f_2 are the first and second component of the flux function (Equation 263). The eigenvalues for this system, λ, are the solution to

$$\begin{vmatrix} u-\lambda & q \\ 0 & -\lambda \end{vmatrix} = 0$$

or $-\lambda(u-\lambda) = 0$, which has two solutions given by $\lambda = 0$ or $\lambda = u$. Note that without knowledge of the sign of u it is impossible to order these eigenvalues in the standard increasing fashion i.e.

$$\lambda^1 < \lambda^2 < \ldots < \lambda^{p-1} < \lambda^p < \ldots < \lambda^n$$

Therefore we will assume (and as required for part (c) of this problem) that $u(x) > 0$ for all x in our domain and then a proper ordering is given by $\lambda^1 = 0$ and $\lambda^2 = u$. The corresponding eigenvector, when $\lambda = 0$ is given by solving

$$\begin{bmatrix} u & q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1^1 \\ r_2^1 \end{bmatrix} = 0\,.$$

which simplifies to a single equation of

$$ur_1^1 + qr_2^1 = 0$$

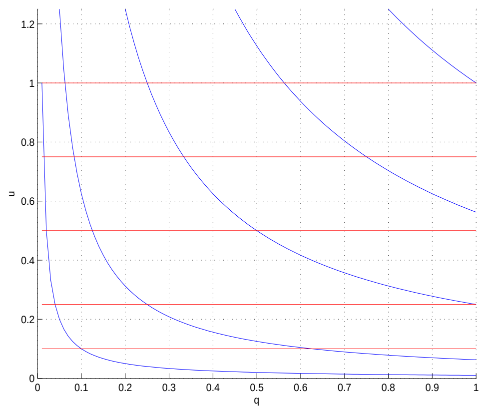

Figure 7: A graphical plot of the one and two integral curves in the q-u plane for Problem 13.11 Part b. The blue curves are 1-integral curves or equivalently 1-Hugoniot curves, while the red curves (horizontal lines) are 2-integral curves or equivalently 2-Hugoniot curves. Note I am assuming that $u > 0$ and $q > 0$ are the only physically meaningful domain in these plots.

giving an eigenvector r^1 of

$$r^1 = \begin{bmatrix} -q \\ u \end{bmatrix} . \tag{265}$$

The corresponding eigenvector for $\lambda = u$ is given by the solution to

$$\begin{bmatrix} 0 & q \\ 0 & -u \end{bmatrix} \begin{bmatrix} r_1^2 \\ r_2^2 \end{bmatrix} = 0 .$$

This system implies that the component r_1^2 can be taken arbitrarily and that r_2^2, must be zero. With these assignments the second eigenvector becomes

$$r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{266}$$

Part (b): For the p-th characteristic field to be linearly degenerate one must have

$$\nabla \lambda^p \cdot r^p = 0 .$$

For $p = 1$ in this problem, we have that

$$\nabla \lambda^1 = \nabla 0 = 0$$

and this characteristic field is obviously linearly degenerate. For $p = 2$ we have that

$$\nabla\lambda^2 = \nabla u = \left(\frac{\partial u}{\partial q}, \frac{\partial u}{\partial u}\right) = (0,1)\,,$$

and thus

$$\nabla\lambda^2 \cdot r^2 = (0,1)\cdot(1,0)^T = 0\,,$$

and both fields are linearly degenerate.

To show that the integral curves and the Hugoniot curves are the same we will compute both and show that they are identical. Now the integral curves for the p-th characteristic field are given by the solution to the following ODEs

$$\tilde{v}'(\xi) = \alpha(\xi) r^p(\tilde{v}(\xi))\,.$$

For $p = 1$ these equations become (taking $\alpha = 1$ for simplicity)

$$\begin{bmatrix} \frac{dq(\xi)}{d\xi} \\ \frac{du(\xi)}{d\xi} \end{bmatrix} = \begin{bmatrix} -q(\xi) \\ u(\xi) \end{bmatrix}\,. \tag{267}$$

Integrating each of these two equations we have

$$\begin{aligned} q(\xi) &= Ae^{-\xi} \\ u(\xi) &= Be^{\xi} \end{aligned} \tag{268}$$

Four our integral curve to pass through the point (q_*, u_*) our constants A, and B can be determined, and the 1-integral curves becomes

$$\begin{aligned} q(\xi) &= q_* e^{-\xi} \\ u(\xi) &= u_* e^{\xi}\,. \end{aligned} \tag{269}$$

Combining these two equations to eliminate ξ we see that along the 1-integral curve we must have

$$q = \frac{q_* u_*}{u}\,, \tag{270}$$

or a regular hyperbola in the q-u plane. For the second integral curve we have the following ODE's to solve

$$\begin{bmatrix} \frac{dq(\xi)}{d\xi} \\ \frac{du(\xi)}{d\xi} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\,. \tag{271}$$

Integrating both of these we have

$$\begin{aligned} q(\xi) &= \xi + C \\ u(\xi) &= D\,, \end{aligned} \tag{272}$$

again to pass through the point (q_*, u_*) our constants C, and D can be determined and the above becomes

$$\begin{aligned} q(\xi) &= \xi + q_* \\ u(\xi) &= u_*\,. \end{aligned} \tag{273}$$

Combining these to eliminate ξ we see that these curves represent horizontal lines in the q-u plane.

The 1-Hugoniot loci are given by solving the Rankine-Hugoniot equations given by

$$s(v_* - v) = f(v_*) - f(v)\,, \tag{274}$$

and imposing the fact that in the limit as $v \to v_*$ the 1 Hugoniot loci approach $r^1(q_*)$ and similarly for the 2 Hugoniot loci. For this system, Equation 274 is given by

$$s\left(\begin{bmatrix} q_* \\ u_* \end{bmatrix} - \begin{bmatrix} q \\ u \end{bmatrix}\right) = \begin{bmatrix} u_* q_* \\ 0 \end{bmatrix} - \begin{bmatrix} uq \\ 0 \end{bmatrix}\,,$$

which reduces to the following two equations

$$\begin{aligned} s(q_* - q) &= u_* q_* - uq \\ s(u_* - u) &= 0\,. \end{aligned} \tag{275}$$

From the second equation we have $u = u_*$ or $s = 0$. If we consider the case when $s = 0$, the first equation above becomes

$$u_* q_* = uq$$

which we see is equivalent to the 1-integral wave curve as developed above, thus this solution corresponds to the *one-shock*. If we consider the case where instead $u = u_*$, then the first equation above becomes

$$s(q_* - q) = u(q_* - q)\,.$$

In this equation there are two possible ways to satisfy it. The first is to have $s = u$ and the second to have $q = q_*$. Now the second possibility cannot be

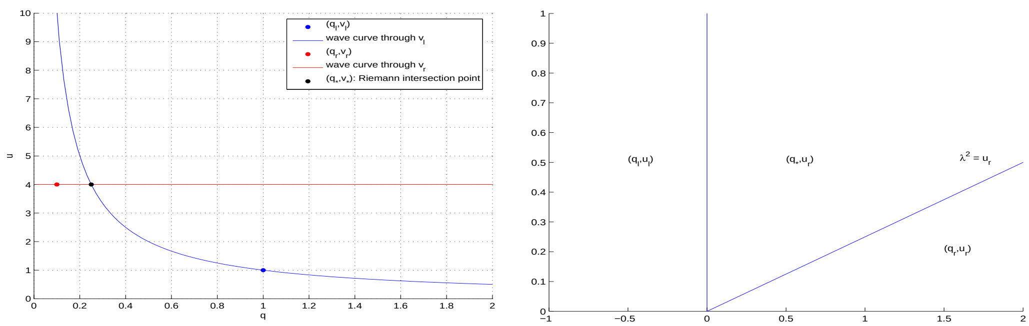

Figure 8: **(Left):** Phase plane solution of a Riemann problem for Problem 13.11. **(Right):** x-t solution of a Riemann problem for Problem 13.11. See the text for additional details.

true since we started this derivation with the assumption that $u = u_*$ and if it were true there would be no state change across this wave. Clearly not a very interesting shock. Assuming then that $s = u$ we see that $q_* - q$ is arbitrary i.e. $q_* - q = \xi$, $q = q_* + \xi$, or a horizontal lines in the q-u plane. These curves are equivalent to the 2-integral waves and thus these solutions correspond to the two-shocks. Plotting each family of integral wave curves (equivalently Hugoniot wave curves) in the q-u plane gives the plot shown in Figure 7.

Part (c): If u_l, $u_r > 0$ then since the integral curves and the Hugoniot loci coincide, given a left state

$$v_l = \begin{bmatrix} q_l \\ u_l \end{bmatrix},$$

and a right state

$$v_r = \begin{bmatrix} q_r \\ u_r \end{bmatrix},$$

through the left (v_l) state draw the 1-rarefaction (equivalently the 1-Hugoniot wave curve) and through the right state (v_r) draw the 2-rarefaction curves (equivalently the 2-Hugoniot wave curve). As an example of this procedure, see Figure **??** (left) for the solution to the Riemann problem in the phase plane. There we have $u_l < u_r$ and $q_l < q_r$. Also in Figure **??** (right) we have drawn the x-t representation for this Riemann problem.

Part (d): If $u_l < 0$ and $u_r > 0$ then the 1-rarefaction and 1-Hugoniot waves through (u_l, q_l) must satisfy $q = \frac{u_l q_l}{u} > 0$ implying that $u < 0$. Since these

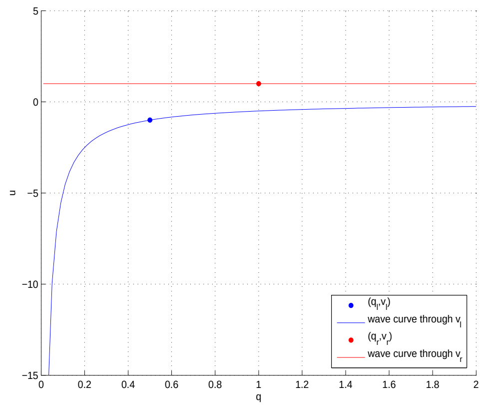

Figure 9: A graphical plot of the one and two integral curves in the q-u plane for Problem 13.11 Part d. Specifically, we have taken left and right states given by $v_l = (0.5, -1)$ and $v_r = (1, 1)$. The blue curves are 1-integral curves or equivalently 1-Hugoniot curves, while the red curves (horizontal lines) are 2-integral curves or equivalently 2-Hugoniot curves. Note that they do not intersect showing that this Riemann problem has no solution.

two curves have no intersections in the q-u plane (see figure 9) no Riemann solution exists.

If $u_l > 0$ and $u_r < 0$ then the 2-rarefaction and Hugoniot curve is below the x-axis. Since $u_l > 0$, however, the 1-rarefaction and Hugoniot curve centered at v_l is *above* the x-axis. This is a symmetric reflection image of the case above. Again since the wave curves don't intersect no Riemann solution exists.

Problem 13.12 (a system from two phase flow)

The system suggested in this problem with $g(v, \phi) = \phi^2$ is given by

$$\begin{aligned} v_t + (v\phi^2)_x &= 0 \\ \phi_t + (\phi^3)_x &= 0 \end{aligned} \tag{276}$$

Note the second equation completely decouples from the first and we can therefore expect that one of the characteristic speeds will be linearly degenerate.

Part (a): Defining conservative variables as

$$q = \begin{bmatrix} v \\ \phi \end{bmatrix}, \tag{277}$$

and flux function $f(q)$ as

$$f(q) = \begin{bmatrix} v\phi^2 \\ \phi^3 \end{bmatrix}, \tag{278}$$

we have a Jacobian given by

$$\frac{\partial f}{\partial q} = \begin{bmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial \phi} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \phi^2 & 2v\phi \\ 0 & 3\phi^2 \end{bmatrix}, \tag{279}$$

which has eigenvalues given by the solution of

$$\begin{vmatrix} \phi^2 - \lambda & 2v\phi \\ 0 & 3\phi^2 - \lambda \end{vmatrix} = 0$$

or

$$(\phi^2 - \lambda)(3\phi^2 - \lambda) = 0$$

giving $\lambda = \phi^2$ or $\lambda = 3\phi^2$. Thus ordering our eigenvalues such that $\lambda^1 < \lambda^2$ we have

$$\lambda^1 \equiv \phi^2 < \lambda^2 \equiv 3\phi^2 . \tag{280}$$

The eigenvector, r^1, of the first characteristic field is given by

$$\begin{bmatrix} 0 & 2v\phi \\ 0 & 2\phi^2 \end{bmatrix} \begin{bmatrix} r^1_1 \\ r^1_2 \end{bmatrix} = 0 . \tag{281}$$

This implies that r^1_1 is arbitrary and $r^1_2 = 0$. Thus the eigenvector is given by

$$r^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{282}$$

Thus this field has no jump in ϕ and only a jump in v. The eigenvector of the second field is given by

$$\begin{bmatrix} -2\phi^2 & 2v\phi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r^2_1 \\ r^2_2 \end{bmatrix} = 0 \tag{283}$$

or

$$-2\phi^2 r^2_1 + 2v\phi r^2_2 = 0 \tag{284}$$

or

$$-\phi r_1^2 + v r_2^2 = 0 \tag{285}$$

which gives for the second eigenvector

$$r^2 = \begin{bmatrix} v \\ \phi \end{bmatrix} \tag{286}$$

The definition of linearly the p-th field being linearly degenerate is if

$$\nabla \lambda^p \cdot r^p = 0 \tag{287}$$

and the p-th field is genuinely nonlinear when

$$\nabla \lambda^p \cdot r^p \neq 0 \tag{288}$$

In our problem we have that

$$\nabla \lambda^1 = (0, 2\phi) \tag{289}$$

and

$$\nabla \lambda^2 = (0, 6\phi) \tag{290}$$

so that

$$\nabla \lambda^1 = (0, 2\phi) \cdot (1, 0) = 0 \tag{291}$$

and

$$\nabla \lambda^2 \cdot r^2 = (0, 6\phi) \cdot (v, \phi) = 6\phi^2 \neq 0 \tag{292}$$

Showing that the first field is linearly degenerate and the second field is genuinely nonlinear.

Part (b): The Hugoniot loci are given by the solutions to the following jump conditions

$$s(q_* - q) = f(q_*) - f(q) \tag{293}$$

which for this problem become

$$\begin{aligned} s(v_* - v) &= v_* \phi_*^2 - v\phi^2 \quad (294) \\ s(\phi_* - \phi) &= \phi_*^3 - \phi^3 \,. \quad (295) \end{aligned}$$

Using the following factorization

$$\phi_*^3 - \phi^3 = (\phi_* - \phi)(\phi_*^2 + \phi\phi_* + \phi_*^2) \,,$$

and assuming that $\phi \neq \phi_*$ the second equation above becomes an equation for s of

$$s = \phi_*^2 + \phi\phi_* + \phi^2 \,. \tag{296}$$

When put into the 1st equation gives

$$(\phi_*^2 + \phi\phi_* + \phi^2)(v_* - v) = v_*\phi_*^2 - v\phi^2 \,.$$

Expanding the left hand side of this equation and canceling common terms we obtain

$$-v\phi_*^2 + v_*\phi_*\phi - v\phi\phi_* + \phi^2 v_* = 0.$$

Solving for v (in terms of ϕ) we first factor v as

$$-\phi_*(\phi + \phi_*)v + \phi(\phi v_* + \phi_* v_*) = 0 \,,$$

which gives for v the following

$$v = \frac{\phi(\phi v_* + \phi_* v_*)}{\phi_*(\phi + \phi_*)} = \frac{\phi v_*(\phi + \phi_*)}{\phi_*(\phi + \phi_*)} = \frac{v_*}{\phi_*}\phi \,. \tag{297}$$

This must be the Hugoniot expression for the *2-waves* since the 1-waves cannot support a jump discontinuity in ϕ. This can be seen by the functional form of r^1 in equation 282 which has a zero in its second component. For the 1-waves no jump in ϕ implies that $\phi = \phi^*$ and the 2nd Rankine-Hugoniot equation 294 is satisfied for all values of s. The first Rankine-Hugoniot equation then requires

$$s(v_* - v) = (v_* - v)\phi^2 \,,$$

and serves to determine s, since if $v_* \neq v$ then $s = \phi^2$ is the shock speed. With all of this information an all shock Riemann solution to this problem looks like that shown in figure 10.

Now the middle state v_m is given by

$$v_m = \frac{v_r}{\phi_r}\phi_l \,, \tag{298}$$

and s^2 given by

$$s^2 = \phi_r^2 + \phi_r\phi_l + \phi_l^2 \,. \tag{299}$$

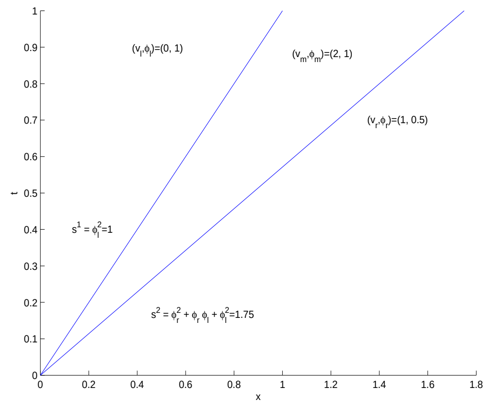

Figure 10: A graphical plot of the x-t solution to a Riemann problem for the system of Problem 13.12. Specifically, we have taken left and right states given by $q_l = (0, 1)$ and $q_r = (1, 0.5)$.

Now we show that these Hugoniot curves are also the *integral curves* for their corresponding families. To show this for the 1-wave we must integrate the following system of ODEs

$$\begin{bmatrix} \frac{dv}{d\xi} \\ \frac{d\phi}{d\xi} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} v(0) \\ \phi(0) \end{bmatrix} = \begin{bmatrix} v_* \\ \phi_* \end{bmatrix}$$

which has a solution given by the following

$$\begin{aligned} v &= \xi + v_* \\ \phi &= \phi_* \,. \end{aligned} \tag{300}$$

We see that the following the 1-integral curve there exists smooth variation in v with *no* jump in ϕ, which is exactly what we found for the 1-Hugoniot curve. For the 2 integral wave curve we must solve

$$\begin{bmatrix} \frac{dv}{d\xi} \\ \frac{d\phi}{d\xi} \end{bmatrix} = \begin{bmatrix} v \\ \phi \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} v(0) \\ \phi(0) \end{bmatrix} = \begin{bmatrix} v_* \\ \phi_* \end{bmatrix} \tag{301}$$

giving as its solution

$$\begin{aligned} v(\xi) &= v_* e^{\xi} \\ \phi(\xi) &= \phi_* e^{\xi} \end{aligned} \tag{302}$$

or eliminating ξ to determine the 2-integral path in the (v, ϕ) plane we obtain

$$\frac{v}{\phi} = \frac{v_*}{\phi_*} \tag{303}$$

exactly as was found before for the 2-Hugoniot loci. Thus the Hugoniot loci and the integral curves are the same for both families and the Riemann problem solution presented a few pages back is then valid in general. The 1-wave maybe described as a contact discontinuity and the 2-wave as a true shock.

Part (c): The Lax-entropy condition for our 2-wave requires that

$$\lambda^2(q_m) > s > \lambda^2(q_r)$$

or

$$3\phi_m^2 > 3\phi_r^2 \,.$$

Now since $\phi_m = \phi_l$, because the 1-wave is a contact discontinuity we have that the Lax-entropy condition relates ϕ_l to ϕ_r by

$$\phi_l > \phi_r \,. \tag{304}$$

Question: How does one proceed to solve the Riemann problem if $\phi_l \leq \phi_r$?

Chapter 15

Problem 15.1

Part (a): From LeVeque Section 15.3.2 (Roe Linearization) assuming a linear path in state space given by

$$q(\xi) = Q_{i-1} + (Q_i - Q_{i-1})\xi \tag{305}$$

then a flux difference can be written as

$$f(Q_i) - f(Q_{i-1}) = \left[\int_0^1 f'(q(\xi))d\xi\right](Q_i - Q_{i-1}) \,. \tag{306}$$

For the p-system (in primitive variables) we have

$$q = \begin{bmatrix} v \\ u \end{bmatrix} \tag{307}$$

with flux given by

$$f(q) = \begin{bmatrix} -u \\ p(v) \end{bmatrix} \tag{308}$$

so the Jacobian $f'(q)$ is then

$$\frac{\partial f}{\partial q} = \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} . \tag{309}$$

The above integral, now defined as $\hat{A}_{i-1/2}$, is explicitly given by

$$\hat{A}_{i-1/2} = \int_0^1 \begin{bmatrix} 0 & -1 \\ p'(v(\xi)) & 0 \end{bmatrix} d\xi \tag{310}$$

Since the expression $p'(v)$ represents the derivative of p with respect to v we can convert this into a derivative with respect to ξ (using the chain rule) as

$$p'(v) = \frac{dp}{d\xi}\frac{d\xi}{dv} = \frac{\frac{dp}{d\xi}}{\frac{dv}{d\xi}} ,$$

and the above integral becomes

$$\hat{A}_{i-1/2} = \int_0^1 \begin{bmatrix} 0 & -1 \\ \frac{dp}{d\xi}\frac{d\xi}{dv} & 0 \end{bmatrix} d\xi \tag{311}$$

This is advantageous since along the linear path $v(\xi) = v_{i-1} + (v_i - v_{i-1})\xi$ the derivative is constant,

$$\frac{dv}{d\xi} = (v_i - v_{i-1}) . \tag{312}$$

So $\hat{A}_{i-1/2}$ becomes

$$\hat{A}_{i-1/2} = \int_0^1 \begin{bmatrix} 0 & -1 \\ \frac{\frac{dp}{d\xi}}{v_i - v_{i-1}} & 0 \end{bmatrix} d\xi , \tag{313}$$

which can be integrated exactly, giving

$$\hat{A}_{i-1/2} = \begin{bmatrix} 0 & -1 \\ \frac{p(\xi=1) - p(\xi=0)}{v_i - v_{i-1}} & 0 \end{bmatrix} , \tag{314}$$

or

$$\hat{A}_{i-1/2} = \begin{bmatrix} 0 & -1 \\ \frac{p(v_i)-p(v_{i-1})}{v_i - v_{i-1}} & 0 \end{bmatrix} \tag{315}$$

as we were to show.

Part (b): If $p(v) = \frac{a^2}{v}$ then our Roe averaged matrix above is given by

$$\begin{aligned} \hat{A}_{i-1/2} &= \begin{bmatrix} 0 & -1 \\ \frac{\frac{a^2}{v_i} - \frac{a^2}{v_{i-1}}}{v_i - v_{i-1}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ \frac{a^2(v_{i-1}-v_i)}{v_i v_{i-1}(v_i - v_{i-1})} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ -\frac{a^2}{v_i v_{i-1}} & 0 \end{bmatrix} \end{aligned} \tag{316}$$

Part (c): WWX: Finish!!! Implement the above in CLAWPACK.

Part (d): This particular Roe solver will require an entropy fix if the underlying is capable of sonic rarefaction. Computing the eigenvalues of the true system we have that

$$\lambda^{1,2} = u \mp \sqrt{-p'(v)}\,. \tag{317}$$

For this specific case with $p(v) = \frac{a^2}{v}$ and $p'(v) = -\frac{a^2}{v^2}$ gives

$$\lambda^{1,2} = u \mp \frac{a}{v}\,, \tag{318}$$

where we can see that if $u = 0$ then a transonic rarefaction would be required. This system requires an entropy fix.

An entropy fix can be added in several ways to this system of nonlinear equations. Such methods, discussed in the text, include: the Harten-Hyman entropy fix, the Harten entropy fix, and the local Lax-Friedrich (LLF) entropy fix. Since this system is relatively simple, with its Riemann solution consisting of only two outgoing waves with a single constant state between, following the suggestion in the book, we will detect when a sonic rarefaction is present and solve for the middle state exactly via integrating along the appropriate wave curve. A procedure for detecting and correcting transonic Riemann problems when using this Roe solver will be presented next.

Consider an interface at $x_{i-1/2}$, separating states Q_{i-1} and Q_i. We first compute some of the characteristic speeds in the left and right states. First compute the left characteristic speed in the left state using

$$\lambda^1_{i-1} = u_{i-1} - \sqrt{-p'(v_{i-1})} = u_{i-1} - \frac{a}{v_{i-1}}\,. \tag{319}$$

Then compute the right characteristic speed in the right state using

$$\lambda^2_{i} = u_{i} + \sqrt{-p'(v_{i})} = u_{i} + \frac{a}{v_{i}}\,. \tag{320}$$

From the computed middle state $q_m = (v_m, u_m)$ (computed with the Roe linearized matrix (above) and the two constant states on either side of our discontinuity at $x_{i-1/2}$) we compute the left and right going characteristics speeds i.e.

$$\lambda^1_m = u_m - \frac{a}{v_m} \tag{321}$$

$$\lambda^2_m = u_m + \frac{a}{v_m}\,. \tag{322}$$

We next check for the possibility of a sonic rarefaction in the first field with the following test

$$\lambda^1_{i-1} < 0 < \lambda^1_m\,. \tag{323}$$

If this condition is true, then we have a transonic rarefaction fan in the 1-wave and we must make some adjustment to the flux, specifically the two fluctuations $\mathcal{A}^-\Delta Q_{i-1/2}$ and $\mathcal{A}^+\Delta Q_{i-1/2}$. In this case we will integrate along the integral curve of r^1, to properly evaluate the flux through the interface at $x = x_{i-1/2}$. From the discussion in the book on centered rarefaction waves given in 13.8.5 for the p-system given, the 1-rarefaction must satisfy

$$\tilde{q}'(\xi) = \frac{r^1(\tilde{q}(\xi))}{\nabla\lambda^1(\tilde{q}(\xi)) \cdot r^1(\tilde{q}(\xi)} \tag{324}$$

now since

$$\lambda^1 = u - \frac{a}{v} \tag{325}$$

so

$$\nabla\lambda^1 = \left(\frac{\partial\lambda^1}{\partial v}, \frac{\partial\lambda^1}{\partial u}\right) = \left(\frac{a}{v^2}, 1\right) \tag{326}$$

WWX: Finish!!!

Problem 15.2

From the discussion in the text the HLL solver is based on approximating the smallest and largest wave speeds and taking the resulting Riemann solution to consist of only *two* waves propagating with speeds $s^1_{i-1/2}$ and $s^2_{i+1/2}$ with a single constant state $\hat{Q}_{i-1/2}$ in between. To satisfy conservation one must have

$$f(Q_i) - f(Q_{i-1}) = s^1_{i-1/2}(\hat{Q}_{i-1/2} - Q_{i-1}) + s^2_{i-1/2}(Q_i - \hat{Q}_{i-1/2}), \tag{327}$$

giving for $\hat{Q}_{i-1/2}$ the expression

$$\hat{Q}_{i-1/2} = \frac{f(Q_i) - f(Q_{i-1}) - s^2_{i-1/2}Q_i + s^1_{i-1/2}Q_{i-1}}{s^1_{i-1/2} - s^2_{i-1/2}}. \tag{328}$$

When we substitute the values $s^1_{i-1/2} = -\frac{\Delta x}{\Delta t}$ and $s^2_{i-1/2} = \frac{\Delta x}{\Delta t}$ as suggested in the text into the expression for $\hat{Q}_{i-1/2}$ we obtain

$$\hat{Q}_{i-1/2} = -\frac{\Delta t}{2\Delta x}(f(Q_i) - f(Q_{i-1})) + \frac{Q_i + Q_{i-1}}{2}. \tag{329}$$

Now the first order Godunov method written in terms of *fluctuations* is given by

$$Q' = Q_i - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}) \tag{330}$$

where I have used the notation $Q' = Q^{n+1}$. The definitions of the fluctuations are given in terms of limited waves by

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}^p_{i-1/2} \tag{331}$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}^p_{i-1/2}. \tag{332}$$

In this problem $m = 2$ and $(\lambda^1)^- = -\frac{\Delta x}{\Delta t}$, and $(\lambda^2)^- = 0$. In the same way we have $(\lambda^1)^+ = 0$, and $(\lambda^2)^+ = \frac{\Delta x}{\Delta t}$. Finally, with definition of the waves given by

$$\mathcal{W}^1_{i-1/2} = \hat{Q}_{i-1/2} - Q_{i-1} \tag{333}$$

$$\mathcal{W}^2_{i-1/2} = Q_i - \hat{Q}_{i-1/2}, \tag{334}$$

we can compute the fluctuations given by

$$\mathcal{A}^+\Delta Q_{i-1/2} = (\lambda^2)^+\mathcal{W}^2_{i-1/2} = \frac{\Delta x}{\Delta t}(Q_i - \hat{Q}_{i-1/2}) \quad (335)$$

$$\mathcal{A}^-\Delta Q_{i+1/2} = (\lambda^1)^-\mathcal{W}^1_{i+1/2} = -\frac{\Delta x}{\Delta t}(\hat{Q}_{i-1/2} - Q_i)\,. \quad (336)$$

Inserting our expansion for $\hat{Q}_{i-1/2}$ (above) the fluctuations becomes

$$\begin{aligned} \mathcal{A}^+\Delta Q_{i-1/2} &= \frac{\Delta x}{\Delta t}\left[Q_i + \frac{\Delta t}{2\Delta x}(f(Q_i) - f(Q_{i-1})) - \frac{1}{2}(Q_i - Q_{i-1})\right] \\ &= \frac{1}{2}(f(Q_i) - f(Q_{i-1})) + \frac{\Delta x}{2\Delta t}(Q_i - Q_{i-1})\,. \end{aligned} \quad (337)$$

In a similar way we have (note the $i + 1/2$ subscript)

$$\begin{aligned} \mathcal{A}^-\Delta Q_{i+1/2} &= -\frac{\Delta x}{\Delta t}\left[-\frac{\Delta t}{2\Delta x}(f(Q_{i+1}) - f(Q_i)) + \frac{1}{2}(Q_{i+1} + Q_i) - Q_i\right] \\ &= \frac{1}{2}(f(Q_{i+1}) - f(Q_i)) - \frac{\Delta x}{2\Delta t}(Q_{i+1} - Q_i)\,, \end{aligned} \quad (338)$$

so that our first order Godunov method (in terms of fluctuations) becomes

$$\begin{aligned} Q_i^{n+1} &= Q_i - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(f(Q_i) - f(Q_{i-1}))\right) \\ &- \frac{\Delta t}{\Delta x}\left(\frac{\Delta x}{2\Delta t}(Q_i - Q_{i-1})\right) \\ &- \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(f(Q_{i+1}) - f(Q_i))\right) \\ &- \frac{\Delta x}{\Delta t}\left(-\frac{\Delta x}{\Delta t}(Q_{i+1} - Q_i)\right) \end{aligned} \quad (339)$$

simplifying this gives

$$\begin{aligned} Q_i^{n+1} &= Q_i - \frac{\Delta t}{2\Delta x}(f(Q_{i+1}) - f(Q_{i-1})) - \frac{1}{2}(Q_i - Q_{i-1} - Q_{i+1} + Q_i) \\ &= Q_i - \frac{\Delta t}{2\Delta x}(f(Q_{i+1}) - f(Q_{i-1})) - Q_i + \frac{1}{2}(Q_{i-1} + Q_{i+1}) \\ &= \frac{1}{2}(Q_{i-1} + Q_{i+1}) - \frac{\Delta t}{2\Delta x}(f(Q_{i+1}) + f(Q_{i-1})) \end{aligned} \quad (340)$$

which is the Lax-Friedrich method and is the requested result.

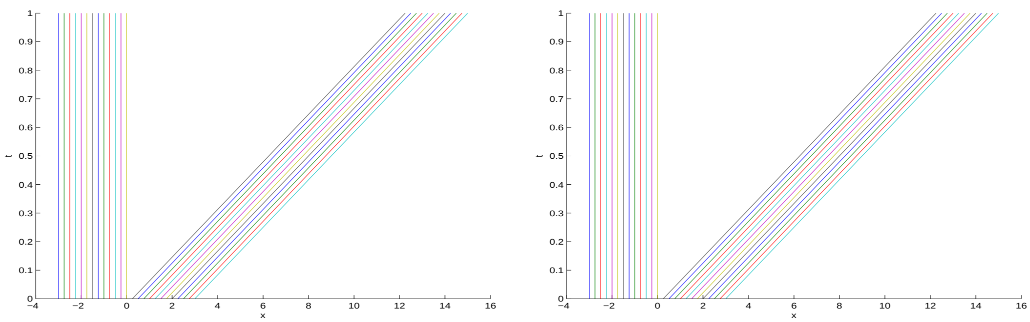

Figure 11: **(Left):** The characteristics for the Riemann problem for Problem 16.1, part a. **(Right):** See the text for additional details.

Chapter 16

Problem 16.1

Our one-dimensional conservation law for this problem is given by

$$q_t + f(q)_x = 0 \tag{341}$$

with $f(q) = q^3$ and various Riemann initial conditions. For this flux we first note that the functional form is *not* convex over the entire real line (because $f''(q) = 6q$ is not one sign) and that the characteristic speeds are given by $f'(q) = 3q^2$.

Part (a): Now for $q_l = 0$ and $q_r = 2$ we have characteristics speeds given by $f'(q_l) = 0$ and $f'(q_r) = 12$, respectively. From these characteristic speeds drawn in figure 11 this problem looks like it would produce a classic centered rarefaction fan at $x = 0$. This is to be expected since on the domain $0 \leq q \leq 2$ our flux function f *is* convex and there is no need to have more complicated (like split) waves. This can be verified from the convex-hull construction. Since $q_l = 0 < q_r = 2$, the convex-hull construction specifies to look for the set of points *above* the set given by

$$\mathcal{R} \equiv \{(q, y) : q_l \leq q \leq q_r \quad \text{and} \quad y \geq f(q)\} .$$

This set of points correspond to the points above the curve $f(q) = q^3$, connecting $(0, 0)$ to $(2, 8)$ is shown in Figure WWX. The fact that the convex hull of the set of points $\mathcal{R}$ is bounded below by the function $f(q)$ implies that the solution of the Riemann problem consists of a single rarefaction separating the states 0 and 2.

Part (b): If $q_l = 2$ and $q_r = -1$, the characteristic speeds are given by $f'(q_l) = 12$ and $f'(q_r) = 3$ respectively. For a convex flux classically we expect a single shock to form. In this case $f(q)$ is non-convex we must use the convex-hull construction to solve this Riemann problem. Since $q_l = 2 > q_r = -1$ the convex-hull construction involves constructing the convex-hull of the following set

$$\mathcal{R} \equiv \{(q, y) : q_l \le q \le q_r \quad \text{and} \quad y \le f(q)\}$$

which is plotted in Figure WWX.

Once this set is constructed our Riemann solution consists of the *largest* y-values. From the above figure one can see that from the point $q_l = 2$ there is a shock to the value q_* from which we have a smooth rarefaction fan to the value $q_r = -1$. The numerical value of q^* is determined by the point at which the slope of the secant line (between the points $(q^*, f(q^*))$ and $(2, f(2))$) equals the tangent to the curve $y = f(q)$ at $q = q^*$. This expression is given by

$$\frac{f(2) - f(q^*)}{2 - q^*} = f'(q^*) \,,$$

which for this equation of state becomes (dropping the asterisk for simplicity)

$$\begin{aligned} \frac{8 - q^3}{2 - q} &= 3q^2 \\ 8 - q^3 &= 6q^2 - 3q^3 \\ 2q^3 - 6q^2 + 8 &= 0 \\ q^3 - 3q^2 + 4 &= 0 \end{aligned}$$

From this last equation we see that trivially $q = -1$ is a root. By long division of the factor $q + 1$ we have the following factorization

$$q^3 - 3q^2 + 4 = (q + 1)(q^2 - 4q + 4) = (q + 1)(q - 2)^2$$

Since $q = -1$ is a root of the above the Riemann solution to the above is given by a single shock from $q_l = 2$ to $q_r = -1$. This shock propagates at the speed given by $f'(q = -1) = 3(-1)^2 = 3$ which in this case corresponds to the same speed as would be calculated with the scalar jump condition

$$s = \frac{f(q_l) - f(q_r)}{q_l - q_r} = \frac{2^3 - (-1)^3}{2 - (-1)} = \frac{8 + 1}{3} = 3 \,,$$

which verifies our results.

Problem 16.2

From the discussion in LeVeque Section 16.1.1, the Buckley-Leverett equation has characteristics speeds given by

$$f'(q) = \frac{2aq(1-q)}{(q^2 + a(1-q)^2)^2}, \tag{342}$$

and we note that for this flux $f'(1) = 0$ and $f'(0) = 0$. For the given initial value we can apply the method of characteristics to a continuous initial condition such as

$$q_0(x) = \left\{ \begin{array}{cc} 1 & x < 0 \\ -\frac{1}{\epsilon}(x-0)+1 & 0 \le x \le \epsilon \\ 0 & x > \epsilon \end{array} \right\} \tag{343}$$

then in the x-t plane each point moves with velocity given by $f'(t)$ and thus ends up at $x = f'(q)t$ at time t. The equal area rule then uses this information to construct an entropy satisfying shock by plotting the position of $tf'(q)$ and drawing a vertical line representing the shock which could cut off equal lobes. In this problem the profile looks like that in Figure 12 (left). When this is rotated counterclockwise by 90 degrees Figure 12 (right) results. Using this rotated plot it is easier to determine the limits of integration in setting up the constraints required by the equal area rule. Now from Figure 12 (right) the areas of the two lobes are given by the following expressions

$$A_I = \int_{q_1}^{q_2} (tf'(q) - x_s)dq = t(f(q_2) - f(q_1)) - x_s(q_2 - q_1) \tag{344}$$

and

$$A_{II} = \int_0^{q_1} (x_s - tf'(q))dq = x_s q_1 - t(f(q_1) - f(0)) = x_s q_1 - tf(q_1) \tag{345}$$

Setting these two areas equal we obtain

$$\begin{aligned} A_I &= A_{II} & (346) \\ t(f(q_2) - f(q_1)) - x_s(q_2 - q_1) &= x_s q_1 - tf(q_1) & (347) \\ tf(q_2) - x_s q_2 &= 0 & (348) \\ x_s &= \frac{tf(q_2)}{q_2} & (349) \end{aligned}$$

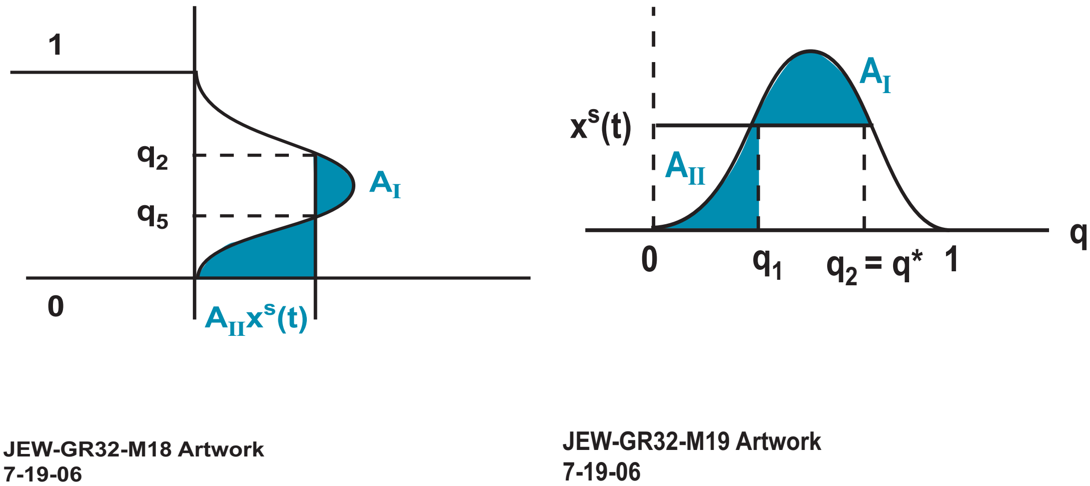

Figure 12: **Left:** The multivalued solution to the Buckley-Leverett equation obtained by the method of characteristics. **Right:** The left figure rotated for ease in determining the limits of the integrals representing the areas of the shaded lobes.

This is an equation for x_s in terms of t but q_2 also depends on t implicitly since it is the right most root of $tf'(q_2) = x_s$. Thus the equation $x_s = \frac{tf(q_2)}{q_2}$ is an implicit equation for either q_2 or x_s. Since for the Buckley-Leverett equation we have that

$$
\begin{aligned}
f'(q) &= \frac{2aq(1-q)}{(q^2+a(1-q)^2)^2} && (350)\\
&= \left(\frac{q^2}{q^2+a(1-q)^2}\right)\frac{2a(1-q)}{q(q^2+a(1-q)^2)} && (351)\\
&= f(q)\left(\frac{2a(1-q)}{q(q^2+a(1-q)^2)}\right) && (352)
\end{aligned}
$$

so the equation $tf'(q) = x_s$ is equivalent to

$$tf(q)\left(\frac{2a(1-q)}{q(q^2+a(1-q)^2}\right) = x_s \tag{353}$$

so

$$t\frac{f(q)}{q} = x_s\left(\frac{q^2+a(1-q)^2}{2a(1-q)}\right) \tag{354}$$

which wen put into the equation $x_s = t\frac{f(q)}{q}$ gives an expression for q^* of

$$\frac{q^2 + a(1-q)^2}{2a(1-q)} = 1 \tag{355}$$

or

$$q^2 + a(1 - 2q + q^2) = 2a - 2aq \tag{356}$$

giving

$$q^2(1+a) = a \tag{357}$$

$$q = \pm\sqrt{\frac{a}{1+a}} \tag{358}$$

Note that this is independent of time and $|q| < 1$, thus the shock location as a function of time is given by

$$x_s(t) = t\left(\frac{\frac{\frac{a}{1+a}}{\frac{a}{1+a}+a\left(1-\sqrt{\frac{a}{1+a}}\right)^2}}{\sqrt{\frac{a}{a+1}}}\right) \tag{359}$$

with

$$f(q) = \frac{q^2}{q^2 + a - 2aq + aq^2} = \frac{q^2}{(1+a)q^2 - 2aq + a} \tag{360}$$

giving

$$\begin{aligned} f(q^*) &= f(\sqrt{\frac{a}{1+a}}) && (361)\\ &= \frac{\frac{a}{1+a}}{a - 2a\sqrt{\frac{a}{a+1}} + a} && (362)\\ &= \frac{\frac{a}{1+a}}{2a(1-\sqrt{\frac{a}{a+1}})} && (363)\\ &= \frac{1}{2a(1-\sqrt{\frac{a}{a+1}})} && (364)\\ &= \frac{1}{2(1+a-\sqrt{a(1+a)})} && (365) \end{aligned}$$

Thus we have that

$$x_s(t) = \frac{t}{2\left(1+a-\sqrt{a(1+a)}\right)}\left(\frac{\sqrt{1+a}}{\sqrt{a}}\right) \tag{366}$$

is the expression of the shock location for the Buckley-Leverett equation is a function of t. The Rankine-Hugoniot equation for this shock (between states $(q^*, 0)$ would be

$$f(q^*) - f(0) = f(q^*) = sq^* \tag{367}$$

thus $s = \frac{f(q^*)}{q^*}$ which is the expression in Equation XXX since q_2 is independent of time. The speed s determined by the convex-hull construction method would satisfy

$$s = \frac{f(q^*)}{q^*} \tag{368}$$

since it connects the state q_2^* and 0 and $f(0) = 0$. Now q_2^* is determined by

$$\frac{f(q_2^*)}{q_2^*} = f'(q_2^*) \tag{369}$$

Given by

$$q_2^* = \sqrt{\frac{a}{1+a}} \tag{370}$$

the same as before. Solving the Riemann problem for the Buckley-Leverett equation using the convex-hull construction requires the location of the point q^* such that

$$\frac{f(q^*)}{q^*} = f'(q^*) \tag{371}$$

or

$$f(q^*) = q^* f'(q^*) \tag{372}$$

or (dropping the asterisk) gives

$$\frac{q^2}{q^2 + a(1-q)^2} = \frac{q^2(2aq(1-q))}{(q^2 + a(1-q)^2)^2} \tag{373}$$

or

$$q^2 + a(1-q)^2 = 2a(1-q) \tag{374}$$

or

$$\begin{aligned} q^2 + a - 2aq + aq^2 &= 2a - 2aq \\ q^2(1+a) &= a \\ q &= \pm\sqrt{\frac{a}{1+a}} \end{aligned} \tag{375}$$

which taking the plus sign gives

$$q^* = \sqrt{\frac{a}{1+a}} < 1 \tag{376}$$

Problem 16.3

LeVeque Equation 16.4 is Oleinik's entropy condition is given by

$$\frac{f(q) - f(q_l)}{q - q_l} \geq s \geq \frac{f(q) - f(q_r)}{q - q_r} \quad \forall q \quad \text{between} q_l \quad \text{and} \quad q_r \tag{377}$$

In the Buckley-Leverett equation $q_l = q^*$ and $q_r = 0$, where q^* is determined by the convex-hull construction and

$$s = \frac{f(q^*)}{q^*}$$

Thus the Oleinik entropy condition becomes

$$\frac{f(q) - f(q^*)}{q - q^*} \geq \frac{f(q^*)}{q^*} \geq \frac{f(q)}{q} \quad \text{for} \quad 0 \leq q \leq q^* \tag{378}$$

Lets assume to be proven incorrect by deriving a contradiction to Oleinik's entropy condition) that the entropy satisfying shock in the Buckley-Leverett equation had its left state such that

$$q_l > q^* = \sqrt{\frac{a}{1+a}}$$

Then Oleinik's entropy condition requires that

$$\frac{f(q) - f(q_l)}{q - q_l} \geq \frac{f(q_l)}{q_l} \geq \frac{f(q)}{q} \tag{379}$$

The first inequality is equivalent to

$$f(q)q_l - f(q_l)q_l \leq f(q_l)(q - q_l) \tag{380}$$

or

$$\frac{f(q)}{q} \leq \frac{f(q_l)}{q_l}$$

or the second inequality. This inequality using the Buckley-Leverett flux is equivalent to

$$\frac{q}{q^2 + a(1-q)^2} \leq \frac{q_l}{q_l^2 + a(1-q_l)^2} \tag{381}$$

Here $q_l > \sqrt{a}1 + a = q^*$. Considering the function $g(q) = \frac{q}{q^2+a(1-q)^2}$ as a function of q on the bounded interval $0 \leq q \leq q_l$ has $g(0) = 0$ with $g(q_l) = \frac{q_l}{q_l^2+a(1-q_l)^2}$ giving

$$\begin{aligned} g'(q) &= = \frac{1}{q^2 + a(1-q)^2} - \frac{q(2q + 2a(1-q)(-1))}{q^2 + a(1-q)^2} \qquad (382) \\ &= \frac{-q^2 + a(1 - 2q + q^2) + 2aq - 2aq^2}{q^2 + a(1-q)^2} \qquad (383) \end{aligned}$$

Problem 16.4

We will assume that f is concave (the situation where f is convex is similar). The definition of concave requires that $f''(q) < 0$ throughout the range of allowable q. Since the *second* derivative is all of one sign the first derivative cannot have a maximum or a minimum over the range of allowable q's. This implies that whatever *sign* $f'(q)$ is (either positive or negative)

Problem 16.5

LeVeque equation 16.13 are two un-coupled Burger's equations or

$$\begin{aligned} u_t + \frac{1}{2}(u^2)_x &= 0 \qquad (384) \\ v_t + \frac{1}{2}(v^2)_x &= 0 \qquad (385) \end{aligned}$$

with data given by LeVeque Equation 16.17 or

$$q_l = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{386}$$

Part (a): The solution to this Riemann problem in the state space consists of connecting q_l to q_r along an entropy satisfying 1-wave from q_l and 2-waves from q_r. Since for this system of eigenvectors change if the state trajectory crosses the line $u = v$ our 1 and 2 wave definitions change depending on which side of the line $u = v$ we fall. Plotting the points q_l and q_r in the u-v plane we can draw r^1 and r^2 vectors. Considering what happens to the individual states u and v can help determine what happens in the u-v plane. The Riemann problem for u has left and right states given by 2 and 0 which corresponds to a right going shock traveling at a speed given by

$$s = \frac{1}{2}(2 + 0) = 1 \tag{387}$$

The Riemann problem for v has left and right states given by 0 and 2 and correspond to a rarefaction fan with solution

$$v(x,t) = \begin{cases} 0 & \frac{x}{t} < 0 \\ \frac{x}{t} & 0 \le \frac{x}{t} \le 2 \\ 2 & \frac{x}{t} > 2 \end{cases} \tag{388}$$

Thus since the

$$q = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \frac{x}{t} < 0 \\ \begin{pmatrix} 2 \\ x/t \end{pmatrix} & 0 < \frac{x}{t} < 1 \\ \begin{pmatrix} 0 \\ x/t \end{pmatrix} & 1 < \frac{x}{t} < 2 \\ \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \frac{x}{t} > 2 \end{cases} \tag{389}$$

This path in the u-v plane with parameter $\xi = \frac{x}{t}$ is given by

$$q(\xi) = \begin{cases} \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \xi < 0 \\ \begin{pmatrix} 2 \\ \xi \end{pmatrix} & 0 < \xi < 1 \\ \begin{pmatrix} 0 \\ \xi \end{pmatrix} & 1 < \xi < 2 \\ \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \xi > 2 \end{cases} \tag{390}$$

Chapter 17

Problem 17.1

The problem in Section 17.2 is

$$q_t + \bar{u}q_x = -\beta q \tag{391}$$

with β constant. This is modified by letting $\beta = \beta(x)$.

Then a second order un-split method can be derived by way of the Lax-Wendroff (Taylor series methods) approach. As such, consider our unknowns at time $t + \Delta t$ by expanding it in a Taylor series as follows

$$q(x, t + \Delta t) = q(x, t) + \Delta t q_t(x, t) + \frac{\Delta t^2}{2} q_{tt}(x, t) + O(\Delta t^3) \tag{392}$$

Then using our equation above we have that

$$q_t = -\beta(x)q - \bar{u}q_x \tag{393}$$

so that

$$\begin{aligned} q_{tt} &= -\beta(x)q_t - \bar{u}q_{tx} && (394) \\ &= -\beta(x)(-\beta(x)q - \bar{u}q_x) - \bar{u}\left(\frac{\partial}{\partial x}(-\beta(x)q - \bar{u}q_x)\right) && (395) \\ &= \beta(x)^2 q + \bar{u}\beta(x)q_x + \bar{u}(\beta'(x)q + \beta(x)q_x + \bar{u}q_{xx}) && (396) \\ &= (\beta(x)^2 + \bar{u}\beta'(x))q + (\bar{u}\beta(x) + \bar{u}\beta(x))q_x + \bar{u}^2 q_{xx} && (397) \end{aligned}$$

Thus we have that

$$q_{tt} = (\beta(x)^2 + \bar{u}\beta'(x))q + 2\bar{u}\beta(x)q_x + \bar{u}^2 q_{xx} \tag{398}$$

Then the Lax-Wendroff method will replace the time derivatives in the Taylor series above and replace them with 2nd order accurate centered finite difference. For example, using

$$q_x \approx \frac{Q_{i+1} - Q_{i-1}}{2\Delta x} \tag{399}$$

similarly for $\beta'(x)$ and

$$q_{xx} \approx \frac{Q_{i+1} - 2Q_i + Q_{i-1}}{\Delta x^2} \tag{400}$$

Thus the Lax-Wendroff method method then becomes for this problem we have

$$
\begin{aligned}
Q_i^{n+1} &= Q_i^n + \Delta t \left(-\beta_i Q_i^n - \bar{u} \left(\frac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x} \right) \right) && (401)\\
&+ \frac{\Delta}{2} \left(\beta_i^2 + \bar{u} \frac{\beta_{i+1} - \beta_{i-1}}{2\Delta x} \right) Q_i^n && (402)\\
&+ \frac{\Delta^2}{2} \left(2\bar{u}\beta_i \right) \left(\frac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x} \right) && (403)\\
&+ \frac{\Delta^2}{2} \left(\bar{u}^2 \right) \left(\frac{Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n}{2\Delta x} \right) && (404)
\end{aligned}
$$

For implementation it is useful to have this expression in term of Q_{i-1}^n, Q_i^n, Q_{i+1}^n. This is given by

$$
\begin{aligned}
Q_i^{n+1} & \left(1 - \beta_i \Delta t + \tfrac{\Delta t^2}{2} \beta_i^2 + \tfrac{\Delta t}{2} \bar{u} \left(\tfrac{\beta_{i+1} - \beta_{i-1}}{2\Delta x} \right) - \tfrac{\Delta t^2}{\Delta x^2} \bar{u}^2 \right) Q_i^n && (405)\\
&+ \left(\frac{\Delta t \bar{u}}{2\Delta x} - \frac{\Delta^2 \bar{u}\beta_i}{2\Delta x} + \frac{\Delta t^2 \bar{u}^2}{2\Delta x^2} \right) && (406)\\
&+ \left(-\frac{\Delta t \bar{u}}{2\Delta x} + \frac{\Delta^2 \bar{u}\beta_i}{2\Delta} + \frac{\Delta t^2 \bar{u}^2}{2\Delta x^2} \right) Q_{i+1}^n && (407)
\end{aligned}
$$

Problem 17.2

The Strang splitting procedure is effectively replacing the operator

$$e^{\Delta t(\mathcal{A}+\mathcal{B})} \quad (408)$$

by

$$e^{\frac{1}{2}\Delta \mathcal{A}} e^{\frac{1}{2}\Delta t \mathcal{B}} e^{\frac{1}{2}\Delta t \mathcal{A}} \quad (409)$$

which becomes

$$(I+\frac{1}{2}\Delta t\mathcal{A}+\frac{1}{8}\Delta t^2 A^2+\frac{1}{48}\Delta t^3 A^3+O(\Delta t^4))(I+\Delta t\mathcal{B}+\frac{1}{2}\Delta t^2 B^2+\frac{1}{6}\Delta t^3 B^3+O(\Delta t^4))(I+\frac{1}{2}\Delta t\mathcal{A}+\frac{1}{8}\Delta t^2 A^2 \quad (410)$$

continuing to expand we have

$$I+\Delta t(B+\frac{A}{2}+\frac{A}{2})+\Delta t^2(\frac{A^2}{8}+\frac{1}{2}B^2+\frac{1}{2}BA+\frac{1}{2}AB+\frac{A^2}{4}+\frac{A^2}{8})+\Delta t^3(\frac{1}{8}BA^2+\frac{1}{4}B^2A+\frac{1}{6}B^3+\frac{1}{48}A^3+\frac{1}{16}A^3 \quad (411)$$

or

$$I+\Delta t(A+B)+\frac{\Delta t^2}{2}(A^2+AB+BA+B^2)+\frac{\Delta t^3}{6}(A^3+\frac{3}{4}A^2B+\frac{3}{2}ABA+\frac{3}{4}BA^2+\frac{3}{2}B^2A+\frac{3}{2}AB^2+B^3) \tag{412}$$

To check our algebra, assuming commutativity of the operators above we obtain

$$\frac{1}{6}(A^3+\left(\frac{3}{4}+\frac{6}{4}+\frac{3}{4}\right)A^2B+3AB^2+B^3) = \frac{1}{6}(A^3+3A^2B+3AB^2+B^3) = \frac{1}{6}(A+B)^3 \tag{413}$$

and our results ...

Problem 17.3

Splitting error for Godunov splitting in system 17.4, consider Godunov splitting which approximates the full solution operator

$$e^{\Delta t(A+B)} = e^{\Delta tA}e^{\Delta tB} \tag{414}$$

Since the expansion of $e^{\Delta t(A_B)}$ is given by Equation 17.30 to determine the error we have simply to compute the product of

$$e^{\Delta tA}e^{\Delta tB} = (I+\Delta tA+\frac{\Delta t^2}{2}A^2++\frac{\Delta t^3}{6}A^3+O(\Delta t^4))(I+\Delta tB+\frac{\Delta t^2}{2}B^2++\frac{\Delta t^3}{6}B^3+O(\Delta t^4)) \tag{415}$$

which equals

$$I+\Delta t(B+A)+\Delta t^2\left(\frac{A^2}{2}+AB\frac{A^2}{2}\right)+O(\Delta t^3) \tag{416}$$

Since the second order term in Godunov splitting does not in general equal the second order term in 17.30 which is

$$\frac{\Delta t^2}{2}(A^2+AB+BA+B^2) \tag{417}$$

we have that Godunov splitting for *non-commutable* operators is only first order (Strang splitting is second order). It is important to note that if the operators A and B commute, then since

$$e^{\Delta t(A+B)} = e^{\Delta tA}e^{\Delta tB} = e^{\frac{1}{2}\Delta tA}e^{\frac{1}{2}\Delta tB}e^{\frac{1}{2}\Delta tA} \tag{418}$$

are identities true to all orders the use of Godunov splitting is *not* an approximation and is in fact exact i.e. valid for all orders of Δt.

Problem 17.5

Equation 17.7 is

$$q_t + \bar{u}q_x = -\beta q \tag{419}$$

with $q(0,t) = C$. Now equation 17.44 provides the exact solution (steady state) given by

$$q(x,t) = Ce^{-\frac{\beta}{\bar{u}}x} \tag{420}$$

Part (a): The unsplit method mentioned in the book is given by

$$Q_i^{n+1} = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \Delta t \beta Q_i^n \tag{421}$$

if a numerical steady state is reached then $Q_i^{n+1} = Q_i^n \equiv Q_i$ the above becomes

$$Q_i = Q_i - \frac{\bar{u}\Delta t}{\Delta x}(Q_i - Q_{i-1}) - \Delta t \beta Q_i \tag{422}$$

which simplifies to or

$$Q_i = \frac{Q_{i-1}}{1 + \frac{\Delta x \beta}{\bar{u}}} \tag{423}$$

Since a Taylor expansion of the true solution gives

which agrees with the previous result to $O(\Delta x)$

Part (b): The fractional split method 17.21 which is combined into one scheme is given by LeVeque Eq. 17.21

$$Q_i^{n+1} = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \beta \Delta t Q_i^n + \frac{\bar{u}\beta\Delta t^2}{\Delta x}(Q_i^n - Q_{i-1}^n) \tag{424}$$

Assuming this scheme converges to a steady state time independent distribution such that $Q_i^{n+1} = Q_i^n$ then the above numerical scheme will enforce that (defining $Q_i^n \equiv Q_i$)

$$0 = \left(-\frac{\bar{u}\Delta t}{\Delta x} - \beta \Delta t + \frac{\bar{u}\beta\Delta t^2}{\Delta x} \right) Q_i + \left(\frac{\bar{u}\Delta t}{\Delta x} - \frac{\bar{u}\beta\Delta t^2}{\Delta x} \right) Q_{i-1} + \tag{425}$$

so we have that

$$Q_i = -\frac{\left(\frac{\bar{u}\Delta t}{\Delta x} - \frac{\bar{u}\beta\Delta t^2}{\Delta x} \right)}{\left(-\frac{\bar{u}\Delta t}{\Delta x} - \beta\Delta t + frac\bar{u}\beta\Delta t^2 \Delta x \right)} \tag{426}$$

or

$$Q_i = \frac{Q_{i-1}\frac{\bar{u}\Delta t}{\Delta x}(1-\beta\Delta t)}{\frac{\bar{u}\Delta t}{\Delta x}(1-\beta\Delta t)+\beta\Delta t} \tag{427}$$

dividing the top and bottom of this expression by $\frac{\bar{u}\Delta t}{\Delta x}(1-\beta\Delta t)$ to get

$$Q_i = \frac{Q_{i-1}}{1+\frac{\beta\Delta t}{\frac{\bar{u}\Delta t}{\Delta x}(1-\beta\Delta t)}} \tag{428}$$

or

$$Q_i = \frac{Q_{i-1}}{1+\frac{\beta\Delta x}{\bar{u}(1-\beta\Delta t)}} \tag{429}$$

or

$$Q_i = \frac{Q_{i-1}}{1+\frac{\beta\Delta x}{\bar{u}}(1+\beta\Delta t)+O(\Delta t^2)} \tag{430}$$

Which seems to have a different sign

Problem 17.5

Equation 17.7 is given by

$$q_t + \bar{u}q_x = -\beta q$$

$$q_t + f(q) = \sum_i \Delta x \Psi_{i-1/2}(t)\delta(x - x_{i-1/2}) \tag{431}$$

Following the suggestion in the book by defining

$$\Psi_{i-1/2} = -\frac{\beta}{Q_{i-1}-Q_i} \tag{432}$$

Then our quasi-steady state problem becomes (in this case around $x \approx x_{i-1/2}$)

$$q_t + \bar{u}q_x = \Delta x \frac{\beta}{2}\delta(x - x_{i-1/2})(Q_{i-1}+Q_i) \tag{433}$$

Now the hyperbolic part of this problem has only a single eigenvalue $\lambda = \bar{u}$ and trivial eigenvector $r = 1$. We are asked to describe the hyperbolic part of this equation with the first order wave fluctuation method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^- Q_{i+1/2}) \tag{434}$$

with

$$\mathcal{A}^-\Delta Q_{i-1/2} = (\bar{u})^-\mathcal{W}^1_{i-1/2} \tag{435}$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = (\bar{u})^+\mathcal{W}^1_{i-1/2} \tag{436}$$

Since we assume that $\bar{u} > 0$ we have that

$$(\bar{u})^- = 0 \tag{437}$$
$$(\bar{u})^+ = \bar{u} \tag{438}$$

The method above becomes

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\bar{u}\mathcal{W}_{i-1/2}) \tag{439}$$

with in the scalar case we have

$$\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n \tag{440}$$

But as discussed in the book to determine the waves we decompose the flux difference (minus the flux impulse $\Delta x \Psi_{i-1/2}$) as

$$f(Q_i) - f(Q_{i-1}) - \Delta x \Psi_{i-1/2} = \beta_{i-1/2}$$

or

$$\bar{u}Q_i - \bar{u}Q_{i-1} + \frac{\Delta x}{2}(Q_{i-1} + Q_i) = \beta_{i-1/2} \tag{441}$$

Then the constant of proportionality in the waves $\mathcal{W}_{i-1/2}$ is given by

$$\alpha_{i-1/2} = \frac{\beta_{i-1/2}}{s_{i-1/2}} = \frac{\beta_{i-1/2}}{\bar{u}} = Q_i - Q_{i-1} + \frac{\Delta x}{2\bar{u}}\beta(Q_i + Q_{i-1}) \tag{442}$$

with $\mathcal{W}_{i-1/2} = \alpha_{i-1/2}$. Thus the first order wave propagation formulation gives them in LeVeques fluctuation method (effectively I'm modeling the waves in LeVeques fluctuation method to incorporate the jump discontinuity at $x = x_{i-1/2}$ by their influence on the wave coefficients. So finally, we have

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\bar{u}\alpha_{i-1/2} = Q_i^n - \frac{\Delta t \bar{u}}{\Delta x}(Q_i^n - Q_{i-1}^n + \frac{\Delta x}{2\bar{u}}\beta(Q_i^n + Q_{i-1}^n)) \tag{443}$$

or

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\bar{u}(Q_i^n - Q_{i-1}^n) - \frac{\beta \Delta t}{2}(Q_i^n + Q_{i-1}^n) \tag{444}$$

The first term is the classic upwind method (as expected with only first order wave fluctuation algorithms), the second term is almost the forcing of LeVeque Equation 17.9 but rather than $\frac{1}{2}(Q_i^n + Q_{i-1}^n)$ and Eq. 17.9 is Q_i^n. The numerical steady state is given by

$$Q_i^{n+1} = Q_i^n = Q_i \tag{445}$$

we obtain for this problem

$$0 = (-\frac{\Delta t}{\Delta x}\bar{u} - \frac{\beta \Delta t}{2})Q_i + \left(\frac{\Delta t}{\Delta x}\bar{u} - \frac{\beta \Delta t}{2}\right) Q_{i-1} \tag{446}$$

or

$$Q_i = \frac{\left(\frac{\Delta t}{\Delta x}\bar{u} - \frac{\beta \Delta t}{2}\right) Q_{i-1}}{\left(\frac{\Delta t}{\Delta x}\bar{u} + \frac{\beta \Delta t}{2}\right)} \tag{447}$$

Multiply the top and bottom by $\frac{\Delta x}{\Delta t}\frac{1}{\bar{u}}$ to get

$$Q_i = \frac{\left(1 - \frac{\beta \Delta x}{2\bar{u}}\right) Q_{i-1}}{\left(1 + \frac{\beta \Delta x}{2\bar{u}}\right)} \tag{448}$$

Which is independent of the timestep size.

Problem 17.6

Show that a shock forms whenever $D > \frac{(1-2q_l)^2}{4}$ our

Problem 17.9

LeVeque equation 17.104 is

$$\begin{aligned} u_t + v_x &= 0 \quad (449) \\ v_t + Au_x &= \frac{f(u) - v}{\tau} \quad (450) \end{aligned}$$

Following the prescription in the book the total system in 17.104 is governed by a coefficient matrix like that in 17.105

$$B = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ \left(\frac{\Delta x}{\Delta t}\right)^2 & 0 \end{bmatrix} \tag{451}$$

and B has eigenvalues given by $\pm\sqrt{\lambda} = \pm\frac{\Delta x}{\Delta t}$. The relaxed scheme described in the book thus solves

$$u_t + v_x = 0 \quad (452)$$

$$v_t + \left(\frac{\Delta x}{\Delta t}\right)^2 u_x = 0 \quad (453)$$

which if we solve with a simple upwind method requires

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(V_i^n - V_{i-1}^n) \quad (454)$$

$$V_i^{n+1} = V_i^n - \frac{\Delta t}{\Delta x}\left(\frac{\Delta t}{\Delta x}\right)^2 (U_i^n - U_{i-1}^n) = V_i^n - \left(\frac{\Delta x}{\Delta t}\right)(U_i^n - U_{i-1}^n) \quad (455)$$

This results in values U^* and V^* i.e.

$$U_i^* = U_i^n - \frac{\Delta t}{\Delta x}(V_i^n - V_{i-1}^n) \quad (456)$$

$$V_i^* = V_i^n - \frac{\Delta x}{\Delta t}(U_i^n - U_{i-1}^n) \quad (457)$$

Next applying the relaxation step results in

$$U_i^{n+1} = U^* = U_i^n - \frac{\Delta t}{\Delta x}(V_i^n - V_{i-1}^n) \quad (458)$$

$$V_i^{n+1} = f(U_i^n - \frac{\Delta t}{\Delta x}(V_i^n - V_{i-1}^n)) \quad (459)$$

so we have that

Chapter 18

Problem 18.1

Consider

$$q_t + Aq_x + Bq_y = 0 \quad (460)$$

with

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad (461)$$

to be simultaneously diagonalizable in the sense that a transformation exists that turns this problem into one involving characteristic variables variables in only characteristic directions then the coefficient matrices must commute i.e. $AB = BA$. In this problem the left hand side is given by

$$AB = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix} \tag{462}$$

and

$$BA = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix} \tag{463}$$

which says that they do commute and thus must have the same eigenvectors. Thus A and B can be simultaneously diagonalizable by a common eigenvector matrix R.

$$A = R\Lambda^x R^{-1} \tag{464}$$

$$B = R\Lambda^y R^{-1} \tag{465}$$

To find R compute the eigenvectors of A or B. Since B is somewhat simple lets compute its eigenvalues λ as

$$\left\| \begin{matrix} -\lambda & 2 \\ 2 & -\lambda \end{matrix} \right\| = 0 \tag{466}$$

or

$$\lambda^2 - 4 = 0$$

or $\lambda^{1,2} = \pm 2$. The the first left eigenvector r^1 is given by

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} r^1 \\ r^2 \end{bmatrix} = 0 \tag{467}$$

which gives $r^1 + r^2 = 0$ which gives

$$r^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{468}$$

In a similar way

$$r^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{469}$$

Now

$$R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{470}$$

So the inverse is given by

$$R^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{471}$$

so that

$$\Lambda^x = R^{-1}AR = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \tag{472}$$

and

$$\Lambda^y = R^{-1}AR = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \tag{473}$$