Try: Note Hermite polys have a esp. simple recurrence relation like difference ∴ might be able to be solved like

Legendre: recurrence

$(n+1)P_{n+1} - (2n+1)zP_n + nP_{n-1} = 0$

Tough to solve this due to nature of coeff. Δ eq of Poincaré type.

Put into Sturm Liouville form.

$$\Delta[p(t-1)\Delta y(t-1)] + [q(t) + \lambda r(t)]y(t) = 0$$

* ⇒

~~$(n+2)P_{n+2} - (2n+3)zP_{n+1} + (n+1)P_n = 0$~~

Mult by

$h(n) \ni$

~~$(n+1)h(n) \ldots (2n+3)z\,h(n)$~~

$$h(n)(n+1)P_{n+1} - h(n)(2n+1)zP_n + h(n)nP_{n-1} = 0.$$

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(3.1) $\Delta y + y(t) = e^t$

$y(t+1) = e^t \qquad p(t) \equiv 0$

$\Rightarrow \quad y(t) = e^{t-1}$

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(3.2) (a) $u(t+1) = \frac{t}{t+1} u(t)$

$u(2) = u(1) \frac{1}{1+1}$

$u(3) = u(2) \frac{2}{1+2}$

$= u(1) \frac{1}{1+1} \cdot \frac{2}{1+2}$

$\vdots$

$$u(t) = \prod_{t'=1}^{t-1} \left(\frac{t'}{t'+1} \right)$$

$$= \frac{(t-1)!}{t!} = \frac{1}{t}.$$

(b) $u(t+1) = \frac{3t+1}{3t+7} u(t)$

$u(2) = \frac{3(1)+1}{3(1)+7} u(1)$

$$u(t) = \prod_{t'=1}^{t-1} u(1) \frac{3t'+1}{3t'+7} = u(1) \frac{(3(t-1)+1)(3(t-2)+1) \cdots (3+1)}{(3(t-1)+7)(3(t-2)+7) \cdots (3+7)}$$

$$= \frac{u(1)\,(3t-2)(3t-5)(3t-8)\cdots 13\cdot 10\cdot 7\cdot 4}{(3t+4)(3t+1)(3t-2)(3t-5)\cdots 13\cdot 10}$$

$$= \frac{4\cdot 7\, u(1)}{(3t+4)(3t+1)}$$

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(3.3) (a) $u(t+1) = e^{3t} u(t)$

$u(1) = u(0)$
$u(2) = e^{3\cdot 1} u(1) = u(0) e^{3\cdot 1}$
$u(3) = e^{3\cdot 2} u(2) = e^{3\cdot 2} e^{3\cdot 1} u(0)$
$\vdots$

$$u(t) = \left(\prod_{i=0}^{t-1} e^{3i} \right) u(0)$$

$$= e^{3(0)} e^{3(1)} e^{3(2)} \cdots e^{3(t-2)} e^{3(t-1)} u(0)$$

$$= \exp\left\{ 3\left(1 + 2 + \cdots (t-2) + (t-1)\right) \right\} u(0)$$

$$= \exp\left\{ 3\left[\frac{t(t-1)}{2} \right] \right\} u(0)$$

$$= e^{\frac{3}{2}t(t-1)} u(0)$$

(b) $u(t+1) = e^{\cos 2t} u(t)$

sol $$u(t) = \left(\prod_{s=a}^{t-1} e^{\cos 2s} \right) u(a)$$

$$u(t) = \exp\left\{\sum_{i=a}^{t-1} \cos 2i\right\}$$

But $$\sum_{i=a}^{t-1} \cos 2i = \frac{\sin 2(k - 1/2)}{2\sin(2/2)}\Bigg|_a^t$$

$$= \frac{\sin 2(t - 1/2)}{2\sin 1} - \frac{\sin 2(a - 1/2)}{2\sin 1}$$

$$= \frac{1}{2\sin 1}\left[\sin 2(t - 1/2) - \sin 2(a - 1/2)\right]$$

$t \geqslant a$

As $$\sum \cos at = \frac{\sin a(t - 1/2)}{2\sin(a/2)} \quad \text{pg 27}$$

$$\therefore u(t) = u(a)\exp\left\{\frac{\sin 2(t - 1/2) - \sin 2(a - 1/2)}{2\sin 1}\right\}$$

$t \geqslant a$

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(3.4)

(a) $y(t+1) - 2y(t) = 5$

Sol to homo. $y(t) = A2^t$

Sol to inhomo Assume $y_p = (A2^t)v(t)$

get $y_p = (A2^t) \sum \frac{5}{A2^{t+1}}$ — can't cancel $2^t \sum \frac{5}{A2^t}$

$= \frac{5}{2} \sum 1 = \frac{5}{2} t$

$\therefore y(t) = \frac{5}{2} t + A2^t$

2nd method Assume sol to inhomo full 0th order poly $y_p = B$

$B - 2B = 5$ correct

$-B = 5 \Rightarrow B = -5$

Check $\frac{5}{2} t$

$\frac{5}{2}(t+1) - 5t \neq 5$

(over)

$$y_p = \cancel{A}2^t \sum \frac{5}{\cancel{A}2^{t+1}} = \frac{5 2^t}{2} \sum \left(\frac{1}{2}\right)^t$$

$$= \frac{5}{2}(\cancel{2^t}) \frac{(\cancel{1/2})^t}{\left(\frac{1}{2} - 1\right)} = -\frac{5}{2} \frac{1}{\left(\frac{1}{2}\right)}$$

$$= -5$$

$$\therefore \; y_{sol} = A2^t - 5$$

(b) $y(t+1) - 4y(t) = 3 \cdot 2^t$

Meth I: Homo $y_H(t) = A4^t$

Inhomo: Assume like $y_p(t) = B2^t$

$$2B\cancel{2^t} - 4B\cancel{2^t} = 3\cancel{2^t}$$

$$-2B = 3$$

$$B = -3/2$$

$$\therefore \; y_{total}(t) = A4^t - \frac{3}{2}(2^t)$$

Meth II: $y_{part}(t) = \cancel{A}4^t \sum \frac{3 \cdot 2^t}{\cancel{A}4^{t+1}}$

$= 4^t\left(\frac{3}{4}\right) \sum \left(\frac{1}{2}\right)^t$

$= 4^t(3/4) \frac{(1/2)^t}{(1/2 - 1)}$

$= 2^t\left(\frac{3}{4}/2\right)(-2) = -\frac{3}{2} 2^t$

$\therefore \quad y_{total}(t) = A4^t - \frac{3}{2} 2^t$

(c) $y(t+1) - 5y(t) = 5^t$

Homo: $y(t) = A5^t$

Inhomo: Assume $y_p(t) = (Bt + C)5^t$

$(Bt + B + C)5 \cdot 5^t - 5 \cdot 5^t(Bt + C) = 5^t$

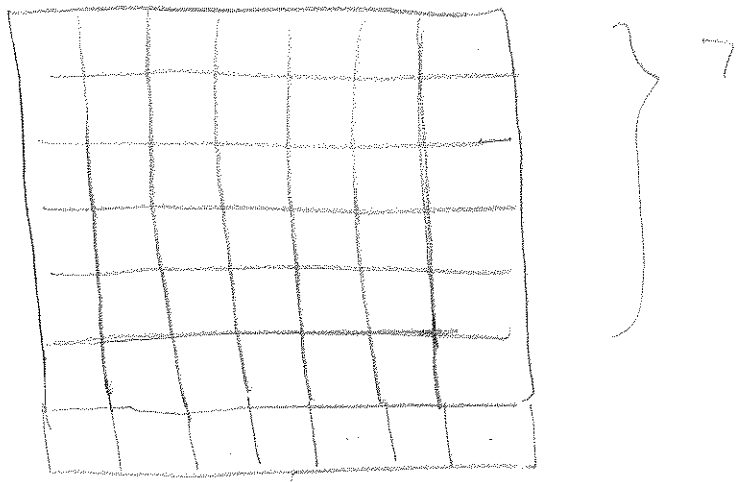
$Bt + B + C - Bt - C = 1/5$

$B = 1/5$

$\therefore \quad y_{tot}(t) = A5^t + \frac{t}{5} \cdot 5^t$

(over)

(3.5) Pg 133 Kelly & Peterson



$$y(t+1) = y(t) + \underbrace{(t+1+t)}_{1\times 1 \text{ squares}}$$

$$+ \underbrace{(t + t-1)}_{2\times 2 \text{ squares}} + \underbrace{(t-1 + t-2)}_{3\times 3 \text{ squares}} +$$

$$+ \cdots + \underbrace{(2+1)}_{(t-1)\times(t-1) \text{ squares}} + \underbrace{(1+0)}_{t\times t \text{ squares}}$$

(1st term) (2nd term)

$$= y(t) + \sum_{k=1}^{t+1} k + \sum_{k=1}^{t} k$$

$$= y(t) + 2\sum_{k=1}^{t+1} k - (t+1)$$

$$= y(t) + 2\sum_{k=1}^{t+1} k^{(1)} - t - 1$$

(over)

(b) $y(t+1) = y(t) + t^2 + 2t + 1$

$y(t+1) - y(t) = t^2 + 2t + 1$

Homo: $y(t) = A$

Nhomo: $y_{part}(t) = A \sum \frac{t^2+2t+1}{E(A)}$

$= \sum (t^2 + 2t + 1)$

$= \sum (t(t-1) + 3t + 1)$

$= \sum (t^{(2)} + 3t^{(1)} + t^{(0)})$

$= \frac{t^{(3)}}{3} + \frac{3t^{(2)}}{2} + \frac{t^{(1)}}{1}$

$= \frac{t(t-1)(t-2)}{3} + \frac{3}{2}t(t-1) + t$

$= \frac{(t^2-t)(t-2)}{3} + \frac{3}{2}(t^2-t) + t$

$= \frac{t^3 - 2t^2 - t^2 + 2t}{3} + \frac{3}{2}(t^2 - t) + t$ (over)

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(3.6) $y(t+1) - 3y(t) = e^t \qquad t \geq 1$

Homo $y(t) = A3^t$

Meth I Inhomo: Assume $y_p(t) = Be^t$

$$Be\cancel{e^t} - 3B\cancel{e^t} = \cancel{e^t}$$

$$B(e-3) = 1$$
$$B = \frac{1}{e-3}$$

Meth II Inhomo: $A3^t \sum \frac{e^t}{A3^{t+1}}$

$$= \frac{\cancel{A}3^t}{3} \frac{(e/3)^t}{\cancel{A}(e/3 - 1)} = \frac{e^t}{(e-3)}$$

$$y_{tot}(t) = A3^t + \frac{e^t}{(e-3)}$$

$$y(1) = 3A + \frac{e}{e-3} = 2$$

$$3A = 2 - \frac{e}{e-3}$$

(over →)

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(3.7) $y(t+1) - \frac{(3t+1)}{(3t+7)} y(t) = \frac{t}{(3t+4)(3t+7)}$

Homo: $y(t+1) = \frac{(3t+1)}{3t+7} y(t)$

$$y(t) = y(1) \prod_{i=1}^{t-1} \left(\frac{3i+1}{3i+7}\right) = y(1) \prod_{i=1}^{t-1} \left(\frac{3i+1}{3(i+2)+1}\right)$$

$$= y(1) \left[\frac{3(1)+1}{3(3)+1} \cdot \frac{3(2)+1}{3(4)+1} \cdot \frac{3(3)+1}{3(5)+1} \cdots \frac{3(t-1)+1}{3(t+1)+1} \right]$$

$$= y(1) \frac{(3(1)+1)(3(2)+1)}{(3(t)+1)(3(t+1)+1)}$$

Inhomo:

$$= Y_{Homo}(t) \sum \frac{\frac{t}{(3(t+1)+1)(3(t+2)+1)}}{E\, Y_{Homo}}$$

$$= \frac{y(1)\, 4 \cdot 7}{(3t+1)(3(t+1)+1)} \sum \frac{\frac{t}{(3(t+1)+1)(3(t+2)+1)}}{\frac{4 \cdot 7\, y(1)}{(3(t+1)+1)(3(t+2)+1)}}$$

over

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(38) $y(t+1) - t\,y(t) = 1 \qquad t \geq 1$

<u>Homo:</u>

$$y(t) = \left(\prod_{k=1}^{t-1} k\right) y(1)$$

$$y(1) = \overbrace{\left(\prod_{k=1}^{0} k\right)}^{=1} y(1) = y(1)$$

$$y(2) = \left(\prod_{k=1}^{1} k\right) y(1) = 1 \cdot y(1)$$

$$\therefore \quad y(t) = (t-1)!\, y(1)$$

<u>Inhomo:</u>

$$y_{part}(t) = (t-1)!\, \cancel{y(t)} \sum \frac{1}{\cancel{y(t)}\, t!}$$

$$= (t-1)! \sum \frac{1}{t!} = (t-1)! \sum_{k=1}^{t-1} \frac{1}{k!}$$

$$\therefore \quad y_{total}(t) = (t-1)! \left[y(1) + \sum_{k=1}^{t-1} \frac{1}{k!} \right]$$

(over)

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(3.9) $y_{n+1}(x) + \frac{x}{n} y_n(x) = \frac{e^{-x}}{n} \qquad n \geq 1$

Homo: $y_{n+1}(x) = -\frac{x}{n} y_n(x)$

$$y_n(x) = \prod_{k=1}^{n-1} \left(-\frac{x}{k}\right) y_1(x)$$

$$= \frac{(-x)^{n-1} y_1(x)}{(n-1)!}$$

Inhomo: $y_n(x) \sum_{n=1}^{N-1} \frac{\left(\frac{e^{-x}}{n}\right)}{y_{n+1}(x)}$

$$= \frac{(-x)^{n-1}}{(n-1)!} \cancel{y_1(x)} \sum_{n=1}^{N-1} \frac{e^{-x}}{n \frac{(-x)^n}{(n!)} \cancel{y_1(x)}}$$

$$y_{part}^n(x) = \frac{(-x)^{n-1}}{(n-1)!} e^{-x} \sum_{k=1}^{n-1} \frac{(k-1)!}{(-x)^k}$$

$$\therefore \quad y_n(x) = \frac{(-x)^{n-1}}{(n-1)!} \left[y_1(x) + e^{-x} \sum_{k=1}^{n-1} \frac{(k-1)!}{(-x)^k} \right]$$

(over)

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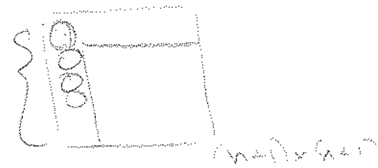
(3.15)

multiplications, n+1 circled cofactors



(a) $Y_{n+1} = (n+1) Y_n + (n+1)$

(n+1) matrices of size n

multiplications w/ cofactors

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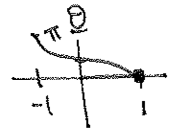

(3.61) extension $|x| < 1$

$$U_0 = \frac{\sin(\cos^{-1} x)}{\sqrt{1-x^2}}$$

$\theta = \cos^{-1} x \quad 0 \leq \theta < \pi$

$\cos\theta = x$

$\sqrt{1-x^2} = \sqrt{\sin^2\theta} = |\sin\theta|$

$$= \frac{\sin(\theta)}{|\sin\theta|} = 1$$

As $\sin\theta > 0$ on this range.

$$U_1 = \frac{\sin(2\cos^{-1} x)}{\sqrt{1-x^2}} = \frac{2\sin(\cos^{-1} x)\cos(\cos^{-1} x)}{\sqrt{1-x^2}}$$

$$= 2x$$

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(3.89) $\Delta t y^2(t) = 1$

$\sum \Rightarrow$

$t y^2(t) = t + C$

$y^2(t) = 1 + C/t$

$y(t) = \pm\sqrt{1 + \frac{C}{t}}$

$$t = (1 + \omega_1 \epsilon + \omega_2 \epsilon^2)\tau$$

$$\ddot{x} = -x - x^3$$

$$\Rightarrow (1 + \omega_1 \epsilon + \omega_2 \epsilon^2)^2 \frac{d^2 x}{d\tau^2} = -x - x^3$$

expand $x = \epsilon x_1 + \epsilon^2 x_2 + \cdots$

$$x^3 = \epsilon^3 x_1^3 +$$

$$\Rightarrow (1 + 2\omega_1 \epsilon + 2\omega_2 \epsilon^2 + \omega_1^2 \epsilon^2 + 2\omega_1 \epsilon^3 \omega_2 + \omega_2^2 \epsilon^4)$$

$O(\epsilon)$

~~$x_1(\tau) = \epsilon \sin \tau$~~

I.C. $\frac{dx}{dt}(t=0) = (1 + \omega_1 \epsilon + \omega_2 \epsilon^2) \frac{dx}{d\tau}\Big|_{\tau=0} = \epsilon.$

~~$x(0) = 0$~~ $\Rightarrow$

$$\frac{dx}{d\tau}(\tau = 0) = \frac{\epsilon}{(1 + \omega_1 \epsilon + \omega_2 \epsilon^2)} = \epsilon\left[1 - \epsilon(\omega_1 + \epsilon\omega_2) + \epsilon^2(\omega_1 + \epsilon\omega_2)^2 \cdots\right]$$

$$\epsilon(\omega_1 + \epsilon\omega_2) \qquad 1 - \epsilon\omega_1 + (-\omega_2 + \omega_1^2)\epsilon^2 + \cdots$$

$$\Rightarrow \frac{dx_1}{d\tau} = 1 \qquad \frac{dx_2}{d\tau} = -\omega_1 \qquad \frac{dx_3}{d\tau} = -\omega_2 + \omega_1^2 \cdots$$

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(3.89) $\Delta t y^2(t) = 1$

$\sum \Rightarrow$

$t y^2(t) = t + C$

$y^2(t) = 1 + C/t$

$y(t) = \pm\sqrt{1 + \frac{C}{t}}$

$$t = (1 + \omega_1 \epsilon + \omega_2 \epsilon^2)\tau$$

$$\ddot{x} = -x - x^3$$

$$\Rightarrow (1 + \omega_1 \epsilon + \omega_2 \epsilon^2)^2 \frac{d^2 x}{d\tau^2} = -x - x^3$$

expand $x = \epsilon x_1 + \epsilon^2 x_2 + \cdots$

$$x^3 = \epsilon^3 x_1^3 +$$

$$\Rightarrow (1 + 2\omega_1 \epsilon^{\to 0} + 2\omega_2 \epsilon^2 + \omega_1^2 \epsilon^2 + 2\omega_1 \epsilon^3 \omega_2 + \omega_2^2 \epsilon^4)$$

$O(\epsilon)$

~~$x_1(\tau) = \epsilon \sin \tau$~~

I.C. $$\frac{dx}{dt}(t=0) = (1 + \omega_1 \epsilon + \omega_2 \epsilon^2) \left.\frac{dx}{d\tau}\right|_{\tau=0} = \epsilon.$$

~~$x(0) = 0 \Rightarrow$~~

$$\frac{dx}{d\tau}(\tau=0) = \frac{\epsilon}{(1 + \omega_1 \epsilon + \omega_2 \epsilon^2)} = \epsilon\left[1 - \epsilon(\omega_1 + \epsilon\omega_2) + \epsilon^2(\omega_1 + \epsilon\omega_2)^2 \cdots\right]$$

$$\epsilon(\omega_1 + \epsilon\omega_2) \qquad 1 - \epsilon\omega_1 + (-\omega_2 + \omega_1^2)\epsilon^2 + \cdots$$

$$\Rightarrow \frac{dx_1}{d\tau} = 1 \qquad \frac{dx_2}{d\tau} = -\omega_1 \qquad \frac{dx_3}{d\tau} = -\omega_2 + \omega_1^2 \cdots$$

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(4.1)

$$v(t+2) - 6v(t+1) + 4w(t+1) - 3v(t) + w(t) = 0$$

$$w(t+2) + w(t+1) + 3v(t+1) - 2w(t) = t3^t$$

Let $u_1(t) = v(t)$

$u_2(t) = v(t+1)$

$u_3(t) = w(t)$

$u_4(t) = w(t+1)$

Let $\vec{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}$

The $\vec{u}(t+1) = \begin{bmatrix} u_1(t+1) \\ u_2(t+1) \\ u_3(t+1) \\ u_4(t+1) \end{bmatrix} = \begin{pmatrix} u_2(t) \\ 6u_2(t) - 4u_4(t) + 3u_1(t) - u_3(t) + 0 \\ u_4(t) \\ -u_4(t) - 3u_2(t) + 2u_3(t) + t3^t \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 6 & -1 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 2 & -1 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t3^t \end{bmatrix}$$

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(4.2) Char eq

$$\lambda^2 + a\lambda + b = 0$$

Companion matrix $A(t) = \begin{bmatrix} 0 & 1 \\ -\frac{b}{1} & -\frac{a}{1} \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ b & \lambda + a \end{vmatrix} = \lambda^2 + a\lambda + b = 0 \quad \text{same}$$

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(4.7)

(a) $$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= 2\begin{bmatrix} & \\ & \end{bmatrix} - 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Does $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ commute?

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underset{=}{\underline{No}} \; 0 \quad 0$$

Try:

$$2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underset{=}{\underline{No}} \; 0 \quad 0$$

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -2 \\ -2 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+1) - 4 = 0$$

$$= \lambda^2 - \lambda - 2 - 4 = 0$$

$$= \lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$\lambda = +3, -2$ $\quad A^t$ sol to $\vec{u}(t+1) = A\vec{u}$ w/ $u(0) = u_0$.

e.vectors of A are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (for -2) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (for $+3$) These are linearly independent

$\therefore$ sol can be written $u(t) = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}[\;]$

(Next pg)

$\lambda = -2, \ \lambda_2 = 3$

$M_0 = I$

$M_1 = (A + 2I) M_0 = A + 2I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

<u>Now:</u> $A^t = \sum_{i=0}^{2-1} c_{i+1}(t) M_i$ w/ c's satisfying

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} \qquad \begin{matrix} c_1(0) = 1 \\ c_2(0) = 0 \end{matrix}$$

$\therefore \quad c_1(t+1) = -2c_1(t) \Rightarrow c_1(t) = (-2)^t = (-1)^t 2^t$

$c_2(t+1) = 3c_2(t) + c_1(t) = 3c_2(t) + (-1)^t 2^t$

Homogen $\rightarrow c_2(t) = k_1 3^t$

Inhomogen try $k_2 (-1)^t 2^t$

$k_2 (-1)^{t+1} 2^{t+1} = 3k_2 (-1)^t 2^t + (-1)^t 2^t$

$\Rightarrow \quad k_2(-2) = 3k_2 + 1 \Rightarrow k_2 = -1/5$

$\therefore$ Sol to $c_2(t)$ is $\quad c_2(t) = \left(-\frac{1}{5}\right)(-1)^t 2^t + K_1 3^t$

$$c_2(0) = -\frac{1}{5} + k_1 = 0$$

$$\therefore \quad c_2(t) = \frac{1}{5}\left[3^t - (-1)^t 2^t\right]$$

Thus $\quad A^t = (-1)^t 2^t I + \frac{1}{5}\left(3^t - (-1)^t 2^t\right)\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (-1)^t 2^t + \left(3^t - (-1)^t 2^t\right)\left(\frac{4}{5}\right) & \frac{2}{5}\left(3^t - (-1)^t 2^t\right) \\ \frac{2}{5}\left(3^t - (-1)^t 2^t\right) & (-1)^t 2^t + \frac{1}{5}\left(3^t - (-1)^t 2^t\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5} 3^t + \frac{1}{5}(-2)^t & \frac{2}{5}\left(3^t - (-2)^t\right) \\ \frac{2}{5}\left(3^t - (-2)^t\right) & \frac{1}{5} 3^t + \frac{4}{5}(-2)^t \end{bmatrix}$$

5

Check: $\underline{t=0}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓ $\underline{t=1}$ $\begin{pmatrix} \frac{12-2}{5} & \frac{2}{5}(5) \\ 2 & \frac{3-3}{5} \end{pmatrix}$ ✓

$\underline{t=2}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Does 2 matrix commute? (Yes everything commutes w/ Identity).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ ✓

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^t = I^t + \binom{t}{1} I^{t-1}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \binom{t}{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 + \binom{t}{3}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^3 + \cdots$$

But $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^3 = 0$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ t & 0 & 0 \\ 0 & t & 0 \end{bmatrix} + \frac{t(t-1)}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \vec{0}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{t(t-1)}{2} & t & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\longleftarrow$

$$\downarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

yes The Above commute.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t + t\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{t-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \vec{0}$$

Now: By induction $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

!

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}$$

(d) $$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Both commute

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^t = I^t + t I^{t-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & t & 0 \end{bmatrix} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}$$

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(4.10) $A = \begin{bmatrix} 3 & 1 \\ -13 & -3 \end{bmatrix}$

$|\lambda I - A| = 0 \Rightarrow \lambda_{1,2} = \pm 2i$

solving $A\bar{v}_1 = \lambda_1 \bar{v}_1$ gives $\bar{v}_1 = \begin{pmatrix} -3+2i \\ 13 \end{pmatrix}$

$A\bar{v}_2 = \lambda_2 \bar{v}_2$ gives $\bar{v}_2 = \begin{pmatrix} -3-2i \\ 13 \end{pmatrix}$ } Not needed

Define λ_i on pg 156 $\quad M_0 = I$

$M_i = (A - \lambda_i I) M_{i-1} \quad 1 \le i \le n$

$\therefore \quad M_0 = I$

$M_1 = (A - 2iI) I = \begin{pmatrix} 3 & 1 \\ -13 & -3 \end{pmatrix} - \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}$

$$= \begin{bmatrix} 3-2i & 1 \\ -13 & -3-2i \end{bmatrix}$$

Then $$A^t = \sum_{i=0}^{n-1} c_{i+1}(t) M_i = \sum_{i=0}^{1} c_{i+1}(t) M_i$$

w/ $\bar{c}$ sol to following $$\bar{c}(t+1) = \begin{bmatrix} \lambda_1 & 0 & \cdots & \\ 1 & \lambda_2 & 0 & \cdots \\ 0 & 1 & \lambda_3 & 0 & \cdots \\ & & \ddots & & \\ & & & 1 & \lambda_n \end{bmatrix} \bar{c}(t)$$

i.c. $\bar{c}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

in our case

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 1 & -2i \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

$c_1(t+1) = 2i\, c_1(t) \Rightarrow c_1(t) = A(2i)^t$

$c_1(0) = 1 \Rightarrow A = 1$

$$C_2(t+1) = (2i)^t - 2i\, C_2(t)$$

$$C_2(t+1) + 2i\, C_2(t) = (2i)^t$$

$\Rightarrow C_2(t) = B(-2i)^t$ for homogeneous pt

inhomo is Assum $C_{2\,inhom}(t) = k(2i)^t$

$$k(2i)(2i)^t + 2i(k(2i)^t) = (2i)^t$$

$$4ik = 1 \qquad k = -\frac{1}{4}i$$

$\therefore \; C_2(t) = B(-2i)^t - \frac{1}{4}i(2i)^t \qquad C_2(0) = 0 \Rightarrow B = \frac{i}{4}$

$$C_2(t) = \frac{i}{4}\left[(-2i)^t - (2i)^t\right]$$

Then $A^t = (2i)^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{i}{4}\left((-2i)^t - (2i)^t\right) \begin{bmatrix} 3-2i & 1 \\ -13 & -3-2i \end{bmatrix}$

$$= \begin{bmatrix} 2^t i^t + \frac{i}{4}\left[(-2)^t i^t - 2^t i^t\right](3-2i) & \frac{(i)^{t+1}}{4}\left((-2)^t - (2)^t\right) \\ -\frac{13}{4}(i^{t+1})\left[(-2)^t - 2^t\right] & 2^t i^t + \frac{(i)^{t+1}}{4}(-1)\left((-2)^t - 2^t\right)(3+2i) \end{bmatrix}$$

Now $(-2)^t - 2^t = 0$ t even

$- 2 \cdot 2^t = -2^{t+1}$

$t = 0$ $= 0$

$t = 1$ $= -4$

$t = 2$ $= 0$

$t = 3$ $= -16$

<u>t even</u> $t = 2k$

$$= \begin{bmatrix} 2^{2k} i^{2k} & 0 \\ 0 & 2^{2k} i^{2k} \end{bmatrix} = \begin{bmatrix} 4^k (-1)^k & 0 \\ 0 & (-4)^k \end{bmatrix}$$

$$= \begin{bmatrix} (-4)^{t/2} & 0 \\ 0 & (-4)^{t/2} \end{bmatrix}$$ t even checked for $t = 2, 4$

<u>t odd</u> $t = 2k+1$ $k = 0, 1, 2, \ldots$ $k = \frac{t-1}{2}$

$$= \begin{bmatrix} 2^t i^t + \frac{(i)^{2k+2}}{\cancel{4}} (-2^{2k+\cancel{2}})(3-2i) & \frac{i^{2k+2}}{4}(-2^{2k+2}) \\ -\frac{13}{\cancel{4}} i^{2k+2} (-2^{2k+\cancel{2}}) & 2^t i^t + \frac{i^{2k+2}}{\cancel{4}} (\cancel{4})(-2^{2k+\cancel{2}})(3+2i) \end{bmatrix}$$

$$= \begin{bmatrix} 2^{k} i(-1)^{k} + (-1)^{k} 2^{2k} (3-2i) & \frac{(-1)^{k} 2^{2k+2}}{4} \\ -13(-1)^{k} 2^{2k} & 2^{k}(-1)^{k} i - (-1)^{k} 2^{2k} (3+2i) \end{bmatrix}$$

$$= \begin{bmatrix} 2^{2k}(-1)^{k}\left[2i + 3 - 2i\right] & (-1)^{k} 2^{2k} \\ 13(-1)^{k+1} 2^{2k} & 2^{2k}(-1)^{k}\left[2i - 3 + 2i\right] \end{bmatrix}$$

$$= \begin{bmatrix} 3(-1)^{\frac{t-1}{2}} 2^{t-1} & (-1)^{\frac{t-1}{2}} 2^{t-1} \\ 13(-1)^{\frac{t+1}{2}} 2^{t-1} & -3(-1)^{\frac{t-1}{2}} 2^{t-1} \end{bmatrix} \quad t \text{ odd} = 1, 3, 5, \ldots$$

7

Check $\underline{t=1}$

$$\begin{bmatrix} 3 & 1 \\ -13 & -3 \end{bmatrix} \checkmark$$

$\underline{t=3}$

$$\begin{bmatrix} -3(4) & -(4) \\ 13(-1)^2 4 & -3(-1)(4) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -4 \\ 52 & 12 \end{bmatrix} \checkmark$$

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$\Delta \log k = \log(k+1) - \log k = f'(c)$

MVT.

$f(x_1) - f(x_2) = f'(c)(x_1 - x_2)$ $\quad x_1 < c < x_2$ $\quad f = \log x$

$\Rightarrow \log k - \log(k+1) = \frac{1}{c}(k-(k+1))$ $\quad k < c < k+1$

$\Rightarrow \Delta \log k = \frac{1}{c} < \frac{1}{k}$; Also $\Delta \log k > \frac{1}{k+1}$

$\frac{n-1}{n-(k+1)} = \frac{n-k-1+k}{n-(k+1)} = 1 + \frac{k}{n-(k+1)} \leq$

$= \leq 1 + k$

10-20-97
253.836

Back

1) Pullups wide 2-1-2, 1½ min rest hoomp
- no help g 10r d 10r
- 20lb dumbbell g 5r d 5r
- ~~15lb g 5~~ d no help g 6r d 5r
- no help inside grip g 10r d 4r. bicept very painfull

1) Lats front wide 2-1-2, 1½ min
- 90lb g 9r d 9r
- ~~90lb g 8r d~~

Stopped bicept tore very painfull

1) Bent dumbell rows
- 45lb g 8r d 8r
- ~~45lb g~~
-
-

Stopped bicept too painfull

1)

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(5.10) (5.11)

$$\int_0^\infty \frac{e^{-t}}{x+t}\,dt = \frac{1}{x}\int_0^\infty \frac{e^{-t}}{1+(t/x)}$$

Large $x \gg 1$

$$\simeq \frac{1}{x}\int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x^k} t^k e^{-t}\,dt \sim \frac{1}{x}\sum_{k=0}^\infty \frac{(-1)^k}{x^k}\underbrace{\int_0^\infty t^k e^{-t}\,dt}_{\Gamma(k+1)}$$

$$= \frac{1}{x}\sum_{k=0}^\infty \frac{(-1)^k\,\Gamma(k+1)}{x^k}$$

(6.5) pg 305 Kelly / Peterson 1

$$\overline{z(t)}\, Ly(t) - y\, \overline{Lz(t)} = \Delta\{p(t-1)\, w[\overline{z(t-1)}, y(t-1)]\}.$$

$$\overline{z(t)}\, Ly(t) = \overline{z}(t)\left[\Delta[p(t-1)\Delta y(t-1)] + q(t)\, y(t)\right]$$

$$= \overline{z}(t)\,\Delta[p(t-1)\Delta y(t-1)] + \overline{z}(t)\, q(t)\, y(t)$$

$$=$$

$$\Delta(uv) = \Delta u \cdot v + Eu\,\Delta v$$

$$Eu = \overline{z}(t-1)$$

$$= \Delta[\overline{z}(t-1)\, p(t-1)\Delta y(t-1)]$$

$$- \Delta\overline{z}(t-1)\, p(t-1)\Delta y(t-1) + q(t)\, y(t)\, \overline{z}(t)$$

$$= \Delta\left[\overline{z}(t-1)\, p(t-1)\Delta y(t-1) - y(t-1)\, p(t-1)\Delta\overline{z}(t-1)\right]$$

$$+ \Delta[y(t-1)\, p(t-1)\Delta\overline{z}(t-1)] - \Delta\overline{z}(t-1)\, p(t-1)\Delta y(t-1)$$

$$+ q(t)\, y(t)\, \overline{z}(t)$$

$$= \Delta\{p(t-1)\, w[\overline{z}(t-1), y(t-1)]\}$$

$$+ \quad y(t)\,\Delta[p(t-1)\Delta\overline{z}(t-1)] + yq\overline{z}$$

= ~~$y(t)\,L\,\bar{z}(t)$~~

$$= y(t)\,L\,\bar{z}(t) + \Delta\left\{p(t-1)\,\omega\left[\bar{z}(t-1),y(t-1)\right]\right\}$$

Note: $\lambda_n = 2\left(1-\cos\left(\frac{n\pi}{4}\right)\right)$

$= 4\sin^2\left(\frac{n\pi}{8}\right) \qquad n = 1, 2, 3.$

Thm 7.1

by Lagrange I

$$\langle Ly, z\rangle = \langle y, Lz\rangle + p(b+1)\,w[\bar{z}(b+1), y(b+1)] - p(a)\,w[\bar{z}(a), y(a)]$$

But. $y, z \in \mathcal{D} \Rightarrow$

$$a_{11}y(a) + a_{12}\Delta y(a) - b_{11}y(b+1) - b_{12}\Delta y(b+1) = 0$$

$$a_{21}y(a) + a_{22}\Delta y(a) - b_{21}y(b+1) - b_{22}\Delta y(b+1) = 0$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} y(a) \\ \Delta y(a)\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix} y(b+1) \\ \Delta y(b+1)\end{pmatrix}$$

eq for $\bar{z}(a)$ & $\bar{z}(b+1)$. put in & subtract.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} y(a) - \bar{z}(a) \\ \Delta y(a) - \Delta\bar{z}(a)\end{pmatrix} =$$

---

want $p(b+1)\,w[\bar{z}(b+1), y(b+1)] = p(a)\,w[\bar{z}(a), y(a)]$

$$\Rightarrow \quad p(b+1)\begin{vmatrix} \bar{z}(b+1) & y(b+1) \\ \Delta\bar{z}(b+1) & \Delta y(b+1)\end{vmatrix} = p(a)\begin{vmatrix} \bar{z}(a) & y(a) \\ \Delta\bar{z}(a) & \Delta y(a)\end{vmatrix}$$

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(7.4) Periodic St-L prob set

p(c) dit B =

transpose of each matrix

$$p(b+1)\begin{vmatrix} \bar{z}(b+1) & \Delta\bar{z}(b+1) \\ y(b+1) & \Delta y(b+1) \end{vmatrix} = p(a)\begin{vmatrix} \bar{z}(a) & \Delta\bar{z}(a) \\ y(a) & \Delta y(a) \end{vmatrix} \quad ??$$

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$$\begin{vmatrix} y(a) & \Delta y(a) \\ \overline{z(a)} & \Delta \overline{z(a)} \end{vmatrix} = \begin{vmatrix} \text{transpose} \\ \text{of this} \end{vmatrix} = w(y(a), \overline{z(a)})$$

$$= -w(\overline{z}(a), y(a)) = 0.$$

$$\begin{vmatrix} y(b+1) & \Delta y(b+1) \\ \overline{z(b+1)} & \Delta \overline{z(b+1)} \end{vmatrix} = 0 = w(y(b+1), \overline{z(b+1)})$$

$$= -w(\overline{z}(b+1), y(b+1))$$

Must also show $p(t)w[\overline{z}(t), y(t)]\Big|_{b+1} = 0$

$\iff w[\overline{z}(t), y(t)]\Big|_{b+1} = 0$

$\alpha y(a) + \beta \Delta y(a) = 0$

$w(\overline{z}(b+1), y(b+1))$

$$= \begin{vmatrix} \overline{z}(b+1) & y(b+1) \\ \Delta \overline{z}(b+1) & \Delta y(b+1) \end{vmatrix} = ?$$

if y & $z \in D$ $\therefore$ $P_y = 0$

$Q_y = 0.$

$\Rightarrow$ from

$Qy = 0$ & $Q\overline{z} = 0$

$\gamma^2 + \delta^2 \neq 0$

$\Rightarrow \gamma y(b+1) + \delta \Delta y(b+1) = 0.$

& $\gamma \overline{z}(b+1) + \delta \Delta \overline{z}(b+1) = 0$

$$\Rightarrow \begin{vmatrix} y(b+1) & \Delta y(b+1) \\ \overline{z}(b+1) & \Delta \overline{z}(b+1) \end{vmatrix} = 0$$

$\Rightarrow$ transpose $= w(y, \overline{z})\Big|_{b+1} = 0$ Also.

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$$\alpha y(a) + \beta \Delta y(a) = 0$$

$\alpha\beta < 0 \Rightarrow \alpha$ neg, β pos
or α pos, β neg

Case (i), 1) $y(a) > 0$ $\quad \Delta y(a) > 0 \quad \Delta y \cdot y > 0$

2) $y(a) < 0$ $\quad \Delta y < 0 \quad \Delta y \cdot y > 0$

$\alpha > 0 \quad \beta < 0$ $\quad$ sim.

$\Rightarrow$

$$y(a)p(a)\Delta y(a) = -\frac{\beta}{\alpha} p(a) \Delta y(a)^2 \geq 0.$$

Pg 320 K/P

$$\alpha y(a) + \beta \Delta y(a) = 0.$$
$$\gamma y(b+1) + \delta \Delta y(b+1) = 0.$$

$\alpha y(a) + \beta y(a+1) - \beta y(a) = 0 \Rightarrow y(a) = -\dfrac{\beta y(a+1)}{\alpha - \beta}$

$$p(a+1)y(a+2) + c(a+1)\,y(a+1) + p(a)y(a) = -\lambda r(a+1)y(a+1)$$

$$'' \quad '' \quad -\frac{\beta p(a)\,y(a+1)}{\alpha - \beta} = -\lambda r(a+1)y(a+1)$$

Pg 320 #7 1

w/ 7.3 ⇒ $\gamma y(b+2) + \delta \Delta y(b+1) = 0$ *

Then let $t = a+1$ in 7.13

$$\Rightarrow \quad p(a+1)y(a+2) + c(a+1)y(a+1) + p(a)y(a) = -\lambda r(a+1)y(a+1)$$

$$7.2 \Rightarrow \quad (\alpha - \beta)y(a) + \beta y(a+1) = 0$$

$$y(a) = -\frac{\beta y(a+1)}{\alpha - \beta} \qquad \text{put in above}$$

$$\Rightarrow \underbrace{\left(c(a+1) + \frac{\beta p(a)}{\beta - \alpha}\right)}_{\tilde{c}(a+1)} y(a+1) + p(a+1)y(a+2) = -\lambda r(a+1)y(a+1)$$

rest o.k. let $t = b+1$ in 7.13

$$\Rightarrow \quad p(b+1)y(b+2) + c(b+1)y(b+1) + p(b)y(b) = -\lambda r(b+1)y(b+1)$$

$$(*) \Rightarrow \quad (\gamma + \delta)y(b+2) - \delta y(b+1) = 0$$

$$y(b+2) = \frac{\delta y(b+1)}{(\gamma + \delta)}$$

$$\Rightarrow \quad p(b)\,\gamma(b) + \Big(\underbrace{c(b+1) + \frac{\delta p(b+1)}{r+\delta}}_{\tilde{c}(b+1)}\Big)\,\gamma(b+1) =$$

$$= -\lambda r(b+1)\,\gamma(b+1)$$

DJ 322 H/P

$a = 0$ $b+1 = 4$ $\Rightarrow b = 3.$

$\beta = 0.$ $c(t) = 0-1-1 = -2$

$\delta = 0.$

$\alpha = 1$

$\gamma = 1.$

$b+1 - (a+1) + 1$

$b - a + 1$

$N = 3 - 0 + 1 = 4$

$$\Delta\left(y(t) - y(t-1)\right) + \lambda y(t) = 0.$$

$$y(t+1) - y(t) - y(t) + y(t-1) = -\lambda y(t)$$

$t = 0, 1, 2, 3, 4$ (a, $a+1$, $a+2$, $a+3 = b$, $b+1$)

1, 2, 3

$$y(t+1) - 2y(t) + y(t-1) = -\lambda y(t)$$

let $t = 1, 2, 3.$

$$y(2) - 2y(1) + y(0) = -\lambda y(1) \qquad (y(0) = 0)$$

$$y(3) - 2y(2) + y(1) = -\lambda y(2)$$

$$y(4) - 2y(3) + y(2) = -\lambda y(3) \qquad (y(4) = 0)$$

$\Rightarrow$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} y(1) \\ y(2) \\ y(3) \end{pmatrix} = -\lambda \begin{pmatrix} y(1) \\ y(2) \\ y(3) \end{pmatrix}$$

$$\begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)\left[(-2-\lambda)^2 - 1\right] - 1\left[(-2-\lambda)\right]$$

$$= (-2-\lambda)\left[4 + 4\lambda + \lambda^2 - 1 - 1\right] = 0$$

$$\Rightarrow \quad 0 = (\lambda+2)(\lambda^2 + 4\lambda + 2)$$

<u>ex 7.1</u> $\Rightarrow \left(2-\sqrt{2},\ \sin\frac{\pi}{4}t\right)$

let $t = 1, 2, 3$ $\quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ As.

$\sin\frac{\pi}{2} = 1$

$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

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7.2. $\alpha y(a) + B\Delta y(a) = 1$

$\to \alpha\left(B y_1(a,\lambda) - \alpha y_2(a,\lambda)\right)$

$+ B\left(B\Delta y_1(a,\lambda) - \alpha\Delta y_2(a,\lambda)\right)$

$= \alpha(B\cdot 1) + B(-\alpha\cdot 1) = 0$ ✓.

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$\Delta y(a) = y(a+1)$ as $y(a) = 0$

$= -y_1(a+1)\left[\frac{y_2(a+1)\,f(a+1)}{p(a)}\right] - y_2(a+1)\sum_{s=a+2}^{b+1} \frac{y_1(s)\,f(s)}{p(a)}$

$= -\frac{y_2(a+1)}{p(a)}\left[\sum_{s=a+1}^{b+1} y_1(s)\,f(s)\right]$

But as $\Delta y_2(a) = 1$ + $y_2(a) = 0$

$\to y_2(a+1) = 1$

$\therefore \Delta y(a) = -\frac{1}{p(a)}\langle y_1, f\rangle$ ✓

Pg 326 #19

$$\Delta y(t-1) = -y_1(t)\sum_{s=a+1}^{t}\frac{y_2(s)f(s)}{p(a)} - y_2(t)\sum_{s=t+1}^{b+1}\frac{y_1(s)f(s)}{p(a)}$$

$\updownarrow$

$$+ y_1(t-1)\sum_{s=a+1}^{t-1}\frac{y_2(s)f(s)}{p(a)} + y_2(t-1)\sum_{s=t}^{b+1}\frac{y_1(s)f(s)}{p(a)}$$

$$= -y_1(t)\frac{y_2(t)f(t)}{p(a)} - [y_1(t) - y_1(t-1)]\cdot\sum_{s=a+1}^{t-1}\frac{y_2(s)f(s)}{p(a)} + y_2(t)\frac{y_1(t)f(t)}{p(a)} - y_2(t)\sum_{s=t}^{b+1}\frac{y_1(s)f(s)}{p(a)}$$

$$+ y_2(t-1)\sum_{s=t}^{b+1}\frac{y_1(s)f(s)}{p(a)}$$

$$= \cancel{-y_1(t)\frac{y_2(t)f(t)}{p(a)}} - \Delta y_1(t-1)\sum_{s=a+1}^{t-1}\frac{y_2(s)f(s)}{p(a)}$$

$$+ \cancel{y_2(t)\frac{y_1(t)f(t)}{p(a)}} - \Delta y_2(t-1)\sum_{s=t}^{b+1}\frac{y_1(s)f(s)}{p(a)}$$

$$\Delta \sum_{s=t}^{b+1} F(s) = \sum_{s=t+1}^{b+1} F(s) - \sum_{s=t}^{b+1} F(s)$$

$$= -F(t)$$

7.1

$$\Delta^2 y(t-1) + \lambda y(t) = 0$$
$$y(0) = 0 \quad y(6) = 0$$

$$\Rightarrow (E - I)^2 y(t-1) + \cdots$$

$$(E^2 - 2E + I) y(t-1) + \cdots = 0$$

$$y(t+1) - 2y(t) + y(t-1) + \lambda y(t) = 0$$

$$y(t+1) - (2-\lambda) y(t) + y(t-1) = 0$$

$$m^2 + (\lambda - 2) m + 1 = 0$$

$$m = \frac{2 - \lambda \pm \sqrt{(\lambda-2)^2 - 4}}{2}$$ Same as example 7.1.

if $|\lambda - 2| > 2$ no pos eigen fn. $m = m_{1,2}$ both $\in \mathbb{R}$

if $|\lambda - 2| < 2$

Let $\lambda - 2 = 2\cos\theta \qquad 0 \le \theta \le \pi$

Then $m = \dfrac{2 - \lambda \pm 2(i)\sin\theta}{2} = \dfrac{2\cos\theta \pm 2i\sin\theta}{2}$

$= \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

Sol: $y(t) = Ae^{i\theta t} + Be^{-i\theta t}$

$y(0) = 0 \Rightarrow A + B = 0 \qquad A = -B$

$y(6) = Ae^{6\theta i} + Be^{-6\theta i} = 0$

$\Rightarrow A\left[e^{6\theta i} - e^{-6\theta i}\right] = 0$

$A\,2i\sin 6\theta = 0$

$6\theta = \pi n \qquad n = 1, 2, 3, \cdots \infty$ ~~[illegible]~~

$\theta_1 = \pi/6,\ \theta_2 = \frac{\pi}{3},\ \theta_3 = \frac{\pi}{2},\ \theta_4 = \frac{2\pi}{3},\ \theta_5 = \frac{5\pi}{6},\ \theta_6 = \pi$

~~Then $y(t) = A\left(e^{i\pi/6 t} - e^{-i\pi/6 t}\right)$~~ $\theta_7 = \frac{7\pi}{6}$

~~$y(t) = 2iA\sin\frac{\pi}{6}t$ just gives 1 efn~~

But then $\lambda = 2 + 2\cos\frac{7\pi}{6} = 2 + 2\cos\frac{5\pi}{6}$. already

$\lambda_n = 2 + 2\cos\theta_n, \qquad n = 1, 2, \ldots, \underline{6}$

~~[illegible]~~

get eifns $\sin\frac{\pi}{6}t,\ \sin\frac{\pi}{3}t,\ \sin\frac{\pi}{2}t,\ \sin\frac{2\pi}{3}t,\ \sin\frac{5\pi}{6}t,$

$\sin\pi t$.

$$2^{-|p||N+\frac{k}{p}|}$$

$$\sum_{p=-\infty}^{\lfloor \frac{-k}{N} \rfloor} (2^N)^p = (2^N)^p \Big|_{-\infty}^{\lfloor \frac{-k}{N} \rfloor}$$

$$\sum \Delta a = (\cancel{1-a}) \sum (a-1) a^n$$

$$\sum a^n = \frac{a}{a-1}$$

$$\sum_{N_1}^{N_2} (a^{n+1} - a^n) = (a-1) \sum$$

$$a^n \Big|_{N_1}^{N_2+1} = (a-1) \sum_{N_1}^{N_2}$$

$$2^{N \cdot \lfloor \frac{-k}{N} \rfloor} = 2^{-k}$$

$$\sum_{p=-\infty}^{\infty} 2^{-|k+pN|}$$

$$\frac{1}{1-r}$$

p

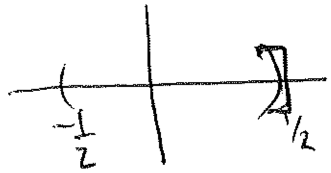
$$p = \frac{-k}{N}$$

$$k = 0$$

$$-\frac{N}{2} \le k < \frac{N}{2} - 1$$

$$-\frac{N}{2} + 1 \le -k \le \frac{N}{2}$$

$$-\frac{1}{2} + \frac{1}{N} \le \frac{-k}{N} \le \frac{1}{2}$$



Pg 335 k/?

$\Delta^2 y(t-1) + \lambda y(t) = 0 \quad y(0) = 0 ; y(6) = 0$

$\xrightarrow{\text{cont}} \quad \frac{d^2y}{dt^2} + \lambda y = 0$

$\Downarrow$

$y(t) = A\sin(\sqrt{\lambda}\, t) + B\cos(\sqrt{\lambda}\, t)$

B.C. $\Rightarrow B = 0 \quad \sqrt{\lambda_n} \cdot 6 = n\pi \qquad n = 1, 2, 3, \ldots$

$\lambda_n = \frac{n^2\pi^2}{36}$.

$\therefore \; y_n(t) = \sin\left(\frac{n\pi}{6} t\right)$

w/ $\lambda_n = \frac{\pi^2 n^2}{36}$ $\quad \infty$ # of eigenvalues

But w/ discrete case

$\exists$ only a finite # of eigvalue

& $\therefore$ we get an alliasing affect.

How well Does discrete sol compair w/ continuous solution?

cont y_n are exactly the same as discrete y_n's up to $n = 6$. Then we get higher modes that are not present in the discrete solutions.

Note that the eigenvalues are not anywhere close to being the same + that if analytically if we continued the discrete solution of $y_n(t) = \sin\left(\frac{\pi}{6} n t\right)$ $\forall\, n \geq 7$ i.e.

7, 8, 9, ... we exactly recover the full spectrum for the continuous problem. Thus on a grid of only 6 data pts we can only hope to "resolve" wave #'s up to $k_n = \frac{\pi n}{6}$ for n up to 6. But we "know" what the wave #'s are for higher n even if they can't be represented on the computer.

(7.2) $\langle Ly, z \rangle = \langle y, Lz \rangle$

$$\| $$

$$\sum_{t=a+1}^{b+1} Ly(t)z(t) = \sum_{t=a+1}^{b+1} \left(\Delta[p(t-1)\Delta y(t-1)] + q(t)y(t)\right) z(t)$$

$$= \sum_{t=a+1}^{b+1} q(t)y(t)z(t) + \sum_{t=a+1}^{b+1} z(t)\,\Delta\left(p(t-1)\Delta y(t-1)\right)$$

$$\|$$

(7.4) $\mathcal{L}\{y(t)\} = 0$ $\qquad a \le t \le b+1$.

$$y(a) = y(b+1)$$

$$* \quad \Delta y(a) = \Delta y(b+1)$$

$$p(a) = p(b+1)$$

By Thm 7.1 + ex. 7.3 periodic S.L prob is self adjoint.

But check

$$p(t)\, w[\bar{z}(t), y(t)] \Big|_a^{b+1} = 0. \qquad y, z \in \mathcal{D}.$$

for $b+1$ $\quad w[\bar{z}(\dots)]\Big|_{b+1} = \begin{vmatrix} \bar{z}(b+1) & y(b+1) \\ \Delta\bar{z}(b+1) & \Delta y(b+1) \end{vmatrix}$

for a $\overset{\text{By B.C.}}{=} \begin{vmatrix} \bar{z}(a) & y(a) \\ \Delta\bar{z}(a) & \Delta y(a) \end{vmatrix} = \bar{z}(a)\big(y(a+1) - \cancel{y(a)}\big) - y(a)\big(\bar{z}(a+1) - \cancel{\bar{z}(a)}\big)$

$$= \bar{z}(a)\, y(a+1) - y(a)\, \bar{z}(a+1)$$

~~$y(a+1) - y(a) = y(b+2) - y(b+1)$~~

Try st forward evaluation of $p(t)\, w[\bar{z}(t), y(t)]\Big|_a^{b+1}$

periodicity of $p(t)$ At at $b+1$ pulls this common factor out

$$\Rightarrow p(a)\Big[\bar{z}(b+1)\Delta y(b+1) - y(b+1)\Delta \bar{z}(b+1)$$

$$- \bar{z}(a)\Delta y(a) + y(a)\Delta \bar{z}(a)\Big] = 0 \quad !!$$

By Periodicity of $y + \bar{z}$

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(7.5) ~~p(t) =~~

$p(t-1) = t$ $\qquad q(t) = -\sin^2 \frac{\pi}{3}t \leq 0 \quad \forall t$

$p(t) = t+1$ $\qquad r(t) = t.$

$a = 2.$ $\qquad b + 1 = 50$

$b = 49.$

$\alpha = 1 ; \beta = -3$ $\qquad \gamma = 0 ; \delta = 1$

$\alpha \cdot \beta = -3 < 0$ ✓ $\quad \gamma \cdot \delta = 0 \geq 0$ ✓

By Thm 7.3 $\lambda > 0$

(7.6) $\Delta^2 y(t-1) + \lambda y(t) = 0$

$a = 0$
$b+2 = 3 \Rightarrow b = 1$
efns $= b - a + 1$
$= 1 - 0 + 1 = 2.$

(a) $(E - I)^2 y(t-1) + \cdots$

$\Rightarrow (E^2 - 2E + I^2) y(t-1) = 0$

$\Rightarrow y(t+1) - 2y(t) + y(t-1) + \lambda y(t) = 0$

$\Rightarrow y(t+1) + (\lambda - 2) y(t) + y(t-1) = 0$

$m^2 + (\lambda - 2) m + 1 = 0$

$m = \dfrac{2 - \lambda \pm \sqrt{(\lambda-2)^2 - 4}}{2}$ if $|\lambda - 2| > 2$ no evalue

Following pg 313 H/P

$\theta_n = \dfrac{n\pi}{3}$ $\lambda_n = 2 - 2\cos\left(\dfrac{n\pi}{3}\right)$ $n = 1, 2, 3$

$y_n(t) = \sin \frac{n\pi}{3} t$ $n = 1, 2.$

(b) $\langle y_1, y_2 \rangle = \displaystyle\sum_{s=a+1=1}^{b+1=2} \sin\left(\frac{\pi}{3}s\right) \sin\left(\frac{2\pi}{3}s\right) = \sin\frac{\pi}{3}\sin\frac{2\pi}{3} + \sin\frac{2\pi}{3}\sin\left(\frac{4\pi}{3}\right)$

where $\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$

$= 0$

(7.7) $\Delta[p(t-1)\Delta y(t-1)] + [q(t) + \lambda r(t)]y(t) = 0$

$a \le t \le b+2$

Finish ~~$\alpha y(a) + \beta \Delta y(a) = 0$~~

~~$\gamma y(b+2) + \delta \Delta y(b+1) = 0$~~

$Py = 0 \quad Qy = 0$

$p(b+1)\det A = p(a)\det B \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Now: if λ is an eigenvalue it must be real

want to show $y^*(t) = y(t)$

~~$\Delta[p^*(t-1)$~~ $\Delta[p(t-1)\Delta y^*(t-1)] + [q(t) + \lambda r(t)]y^* = 0$

As p, q, r are all real valued on int of t pg 311 K/P

subtract from 7.1

$\Rightarrow \Delta[p(t-1)\Delta[y^*(t-1) - y(t-1)] + [q(t) + \lambda r(t)](y^*(t) - y(t)) = 0$

$= \Delta[p(t-1)\Delta[v(t-1)]] + [q(t) + \lambda r(t)]v(t) = 0$

$v(t) = y^*(t) - y(t) \qquad$ B.C. for $v(t)$ are

$Pv = Py^* - Py$

Pg 336 k/P

(7.13)

$w(t) = t^2 - 4t + 3$. $\quad t \in [a+1, b+1] = [1, 2]$

$$w(t) = \sum_{k=1}^{2} c_k y_k(t)$$

$$\Rightarrow \quad c_k = \frac{\langle w, y_k \rangle}{\langle y_k, y_k \rangle}$$

$$c_1 = \sum_{t=1}^{2} (t^2 - 4t + 3) \sin\left(\frac{\pi}{3} t\right) = \sin\left(\frac{\pi}{3}\right)(0) + \sin\left(\frac{2\pi}{3}\right)(4 - 8 + 3)$$

$$= -\sin\left(\frac{2\pi}{3}\right)$$

$$c_2 = \sum_{t=1}^{2} (t^2 - 4t + 3) \sin\left(\frac{2\pi}{3} t\right) = -\sin\frac{4\pi}{3} = \sin\frac{\pi}{3}$$

$$\therefore \quad w(t) = t^2 - 4t + 3 = -\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{3} t\right)$$

$$+ \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3} t\right) \qquad t = 1, 2.$$

How does RHS(t) look compared to LHS(t) $\forall$ t continuous in [1, 2]

& what if we take more points?

PS 343 Kelley + Peterson

$$\sum_{m}^{n-1} a_k$$

$$\sum y(t)\Delta z(t) = y(t)z(t) - \sum E z(t)\Delta y(t)$$

$$+ f_r(t, y(t), \Delta y(t-1))\eta(t-1)\Big|_{a+1}^{b+2} - \sum_{a+1}^{b+1} \Delta f_r(t, y(t), \Delta y(t-1))\eta(t)$$

$$= f_r(b+2, y(b+2), \Delta y(b+1))\eta(b+1) - f_r(a+1, y(a+1), \Delta y(a))\eta(a)$$

$$- \sum_{a+1}^{b+1} \Delta f_r(\cdots)\eta(t)$$

Ideas:

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{ik\Delta x j} e^{-ik'\Delta x j} = \delta_{kk'}$$

$k = k'$

$$= \frac{1}{N}\sum_{j=0}^{N-1} 1 = 1 \quad \checkmark$$

$$\sum a^n = \frac{a^n}{a-1}$$

$k \neq k'$

$$\frac{1}{N}\sum_{j=0}^{N-1}\left[e^{i(k-k')\Delta x}\right]^j = \frac{1}{N}\,\frac{e^{i(k-k')\Delta x j}}{\left(e^{i(k-k')\Delta x}-1\right)}\Bigg|_0^N$$

$$\Delta x = \frac{2\pi}{N}$$

$$= \frac{1}{N}\,\frac{\left(e^{i2\pi(k-k')}-1\right)}{\left(e^{i(k-k')\Delta x}-1\right)} = 0 \quad \checkmark$$

get $$Y_k(j) = \frac{e^{ik\Delta x j}}{\sqrt{N}}$$

Then $$\sum_{j=0}^{N-1} Y_k(j)\,\overline{Y_{k'}(j)} = \delta_{kk'}$$

get Diff eq for $y_k(j) = \frac{e^{ik\Delta x j}}{\sqrt{N}}$

$$\Rightarrow \quad y_k(t) = \frac{e^{ik\Delta x t}}{\sqrt{N}}$$

$$\Delta^2 y_k(t-1) =$$

$$= y_k(t+1) - 2y_k(t) + y_k(t-1) = \frac{1}{\sqrt{N}} e^{ik\Delta x t}\left[e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right]$$

$$= y_k(t)\left[2\cos(k\Delta x) - 2\right]$$

$$= 2y_k(t)\left[\cos(k\Delta x) - 1\right] \qquad \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$= -4\sin^2\left(\frac{k\Delta x}{2}\right) y_k(t)$$

$\Rightarrow$ $y_k(t)$ satisfies

$$\Delta^2 y_k(t-1) + \underbrace{4\sin^2\left(\frac{k\Delta x}{2}\right)}_{\text{"}\lambda_k\text{"}} y_k(t) = 0$$

Note: $y_k(t) = \frac{e^{ik\Delta x j}}{\sqrt{N}}$ is the finite difference equation solution to.

$$\Delta^2 y_k(t-1) + 4\sin^2\left(\frac{k\Delta x}{2}\right) y_k(t) = 0$$

+ B.C. ??

$$\therefore\ T_n(x) = \tfrac{1}{2}\left(x + i\sqrt{\quad}\right)^n + \tfrac{1}{2}\left(x - i\sqrt{\quad}\right)^n \qquad |x| < 1$$

$$T_2(x) = \tfrac{1}{2}\left(x^2 + 2i\sqrt{\quad} - (1-x^2)\right)$$

$$+ \tfrac{1}{2}\left(x^2 - 2i\sqrt{\quad} - (1-x^2)\right)$$

$$= x^2 - (1-x^2) = 2x^2 - 1.$$

$T_{n+2} - 2x T_{n+1} + T_n = 0$ $|x|<1$

$\lambda^2 - 2x\lambda + 1 = 0$

$\lambda = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$

$= x \pm i\sqrt{1-x^2}$

$\therefore T_n = A(x+i\sqrt{1-x^2})^n + B(x-i\sqrt{1-x^2})^n$

$T_0 = 1$

$\Rightarrow A + B = 1$

$T_n(x) = A(x+i\sqrt{\;})^n + (1-A)(x-i\sqrt{\;})^n$

$T_1 = A(x+i\sqrt{\;}) + (1-A)(x-i\sqrt{\;}) = x$

$= A[x+i\sqrt{\;} - x + i\sqrt{\;}] = x-(x-i\sqrt{\;})$

$= i\sqrt{\;}$

$\Rightarrow A = \frac{i\sqrt{\;}}{2i\sqrt{\;}} = \frac{1}{2}$