pg 5 CLR

(1.1-1)

31 41 59 26 41 58

31 41 59 26 41 58

31 41 59 26 41 58

26 31 41 59 41 58

26 31 41 41 59 58

26 31 41 41 58 59

(1.1-2) ↘ rather than ↗

~~Change~~

Given 31 41 59 26 41 58

41 31 59 26 41 58

59 41 31 26 41 58

59 41 31 26 41 58

59 41 41 31 26 58

59 58 41 41 31 26

Alg would cheng while $i > 0$ and $A[i] > key$

to while $i > 0$ and $A[i] < key$

(1.1-3) $A = \langle a_1, a_2, \ldots, a_n \rangle$

Linear search:

```
i ← 1
while i ≤ length(A) do
    if ( A[i] == key ) break
    else i ← i+1
endwhile
if ( i > length(A) )   i = nil
```

(1.1-4) $A, |A| = n$ $B, |B| = n$

Write pseudocode for the vector C ∋ $C = A + B$ bit wise

```
carry = 0                 /* carry = 0, 1 */
i = 1;
while i ≤ length(A) do
    C(i) = A(i) + B(i) + carry    /* binary add. */
    if ( A(i) == 1 and B(i) == 1 or
         A(i) == 1 and carry == 1 or
         carry == 1 and B(i) == 1 ) then
      carry = 1
endWhile
if carry == 1 then C(n+1) = 1
```

$$\begin{array}{r} 0\,1\,0\,1 \\ +\ 0\,0\,1\,1 \\ \hline 0\,1\,1\,1 \end{array}$$ w/ carry.

Binary Add

$$\begin{array}{r} 1 \\ 1 \\ +\ 1 \\ \hline 11 \end{array} \qquad \begin{array}{r} 1 \\ +\ 1 \\ \hline 10 \end{array}$$

pg 11 CLR

03-27-02 1

1.2-1 selection sort. sorting from smallest to largest

Blank array B

31, 41, 59, 26, 41, 58 26

~~26 31 41 59 41 58~~

~~31~~ ~~31~~,

31, 41, 59, x, 41, 58 26,

x 41, 59, x, 41, 58 26, 31,

x, x, 59, x, 41, 58 26, 31, 41,

x, x, 59, x, x, 58 26, 31, 41, 41,

⋮

Pseudocode:

Find smallest elt in array $\tilde{A}$

put into array B

~~$\tilde{A}$ = A~~

~~For~~ i = 1

```
while i ≤ length(A) do
   # Find smallest elmt in Ã say elt j

   B(i) = A(j)
   A(j) = +∞
   i = i+1
enddo
```

```
c1          i = 1
c2      while i ≤ length(A) do
c3  min    { j = 1st elt in A  /* killed elements excluded */
    over   { min = A(j)
    A.     { loop over all elts of A(j) return min.

              B(i) = A(j)
              A(j) = +∞
              i = i+1

        end do
```

The Best cost and worst case for this algorithm are the same. One does not know apriori that one does not need to search all elts of the sub array A (A w/ some elts deleted).

Thus the complexity would go as

$$c_1 + c_2 n + c_3 \sum_{i=1}^{n} \underbrace{(n-i+1)}_{\text{elts to min over}}$$

$$= c_1 + c_2 n + c_3 n^2 - c_3 \sum_{i=1}^{n} i + c_3 n$$

$$= c_1 + c_2 n + c_3 n^2 - c_3 \frac{n(n+1)}{2} + c_3 n$$

$$= c_1 + c_2 n + \cancel{c_3 n^2}\; c_3 \frac{1}{2}(2n^2 - n^2 - n) + c_3 n$$

$$= c_1 + c_2 n + \frac{n^2 c_3}{2} - \frac{c_3 n}{2} + c_3 n$$

$\therefore \quad T(n) = An^2 + Bn + C = \Theta(n^2)$

(1.2-2) W/o any special assumptions on the input ~~arr~~ array A.
I would say $\underline{\underline{n}}$ elements would need to be searched

~~Best~~ Assuming that ½ of the elts are larger than v + ½ are smaller on average $n/2$ elts will have to be searched.

Best case: v is 1st elt of the array

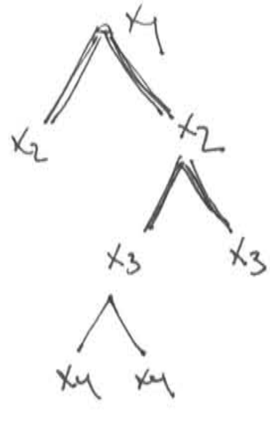
worst case: v is last elt. of the array.

$$T(n) = \Theta(n)$$

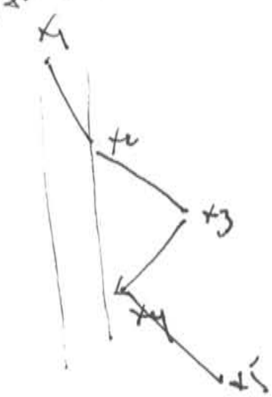
(1.2-3) $A = \langle x_1, x_2, x_3, \ldots, x_n \rangle$ arbitrary sequence. Does A have a repeated element?

$T(n) = n + n-1 + n-2 + \cdots + 1$ for an exhaustive search in the worst case

$$= \frac{n(n+1)}{2}$$



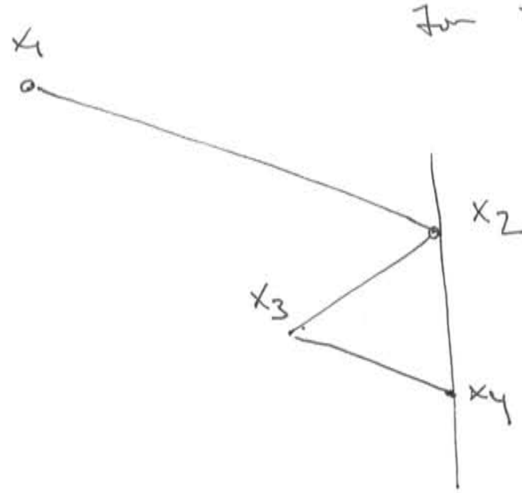
instead of forming a tree from the sequence



consider a # line

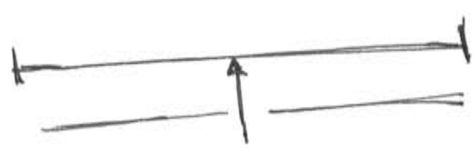
Place the elements, on the page from x_1 to x_n. Thus the tree could look like

For the 4 elt seq $\langle 0, 3, 2, 3 \rangle$



~~[illegible]~~ The vertical line through two pts says that $\exists$ a duplicate entry.

? what is the complexity of Bisection search?



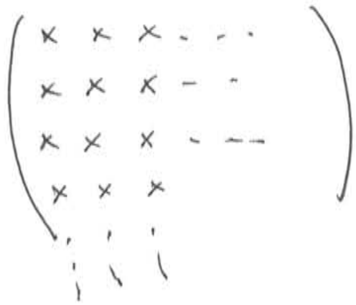
$$\cancel{\sum_{i=1}^{\log_2 n} (1+1+\cdots+1)} =$$

$$1+1+\cdots+1 = \log_2 n$$

Don't see how to do this?

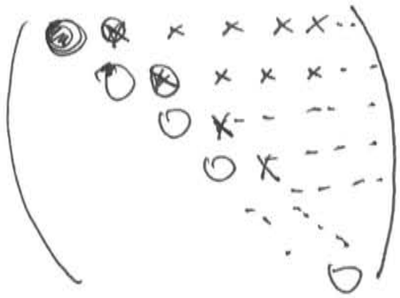
(1.2-3) $\langle x_1, x_2, x_3, \dots, x_n \rangle = A$

$x_i = x_j \ \forall \ i,j$



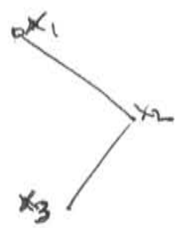
Fill matrix w/ yes/no answers to the question $x_i = x_j \ \forall \ i \neq j$

Thus don't need ~~to~~ to diagonals & matrix is symmetric so



this leads to $n^2 - n - \frac{n^2}{2} = \frac{n^2}{2} - n$

$\log_2 n$



If one could sort these n elts in $n \log n$ time we would be done for in determining the location of a particular element.

PS 11 CLR

1.2-4

$$P(x) = \sum_{i=0}^{n-1} a_i x^i$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Algo #1:

c_1 ~~[illegible]~~ sum = 0

c_2 i = 1

c_3 dowhile $i \le n-1$

$t_i = (i+1)$ sum = sum + $a_i x^i$

enddo

$$T(n) = c_1 + c_2 + c_3(n) + c_4 \sum_{i=1}^{n-1} t_i = c_1 + c_2 + c_3 n + c_4 \sum_{i=1}^{n-1} i+1$$

$$= c_1 + c_2 + c_3 n + c_4(n-1) + c_4\left(\frac{n(n+1)}{2} - n\right)$$

$$= \underline{\underline{\Theta(n^2)}}$$

Alg #2:

~~sum = 0~~ ~~i = n-1~~

~~i = n-1~~ ~~sum = $a_i x$~~

~~dowhile $i \ge 1$~~

~~sum =~~ ~~sum + a_i~~

~~[illegible]~~

c_1 i = n-1

c_2 sum = 0

c_3 do while $i \ge 0$

~~sum =~~ ~~sum + $a_i x$~~ ~~$a_i x$~~

~~enddo~~

$c_4\, t_i$ sum = sumx + a_i

~~enddo~~

t_i i = i-1

enddo

The running time of this second algorithm is

$$T(n) = c_1 + c_2 + c_3(n+1) + c_4 \sum_{i=0}^{n-1} t_i$$

$$+ c_5 \sum_{i=0}^{n-1} t_i$$

But t_i is constant amount of work

$t_i = c \;\therefore\; T(n) = \Theta(n)$

03-28-01

pg 15 CLR

1.3-1

3 41 52 26 38 57 9 49

3 41 52 26 38 57 9 49

3 41 52 26 38 57 9 49

3 41 52 26 38 57 9 49

3 41 52 26 38 57 9 49

3 41 26 52 38 57 9 49

3 26 41 52 9 38 49 57

3 9 26 ~~41~~ 38 41 49 52 57

pg 15 CLR

(1.3-2)

Merge (A, p, q, r) merges sorted arrays A(p, q) + A(q+1, r) into a Blank array B. Note: It would prob. be nice to have the sorted array held in array A but in the interest of time (mine) I'll do this problem like this

```
leftHead = p          /* beginning of "left" array */
rightHead = q+1       /*    "       "  "right"  "   */
newElt = 1
dowhile( leftHead < q+1 + rightHead < r )
    if ( A(lH) ≤ A(rH) )
        B(newElt) = A(lH)
        lH = lH+1
    else
        B(newElt) = A(rH)
        rH = rH+1
    endif
    newElt = newElt+1
end do
```

~~if (leftHead ≤ q+1)~~

```
dowhile (lH < q+1)          } will never be executed if
    B(newElt) = A(lH)       } left array "runs out 1st"
    lH = lH+1
    newElt = newElt+1
enddo

dowhile ( rH < r )          } will never be executed if
    B(newElt) = A(rH)       } right array "runs out 1st"
    rH = rH+1 ; newElt = newElt+1   enddo
```

pg 15 CLR

(1.3-3) $T(n) = n\log_2 n$ when n is a power of 2.

Check $T(2) = 2\log_2 2 = 2$ yes.

Assume $T(n) = n\log_2 n$ for n a power of 2

$n = \{2, 2^2, \ldots, 2^{k-1}, 2^k\}$.

Pr: $T(2^{k+1}) = 2^{k+1}\log_2(2^{k+1})$

Now

$$T(2^{k+1}) = 2T\left(\frac{2^{k+1}}{2}\right) + 2^{k+1} = 2T(2^k) + 2^{k+1}$$

$$= 2\cdot 2^k \log_2 2^k + 2^{k+1}$$

$$= 2^{k+1}[k+1] = 2^{k+1}\log_2 2^{k+1}$$ done!!

pg 15 CLR

(1.3-4) Rec - Insertion sort ~~Alg~~ (A, n)

Rec-Insertion Sort (A, n-1)

Insert A(n) into sorted list.

End.

worst case we have to move all elts, once to place A[n].

$$T(n) = T(n-1) + \Theta(n-1)$$

$$\Delta T(n) = \Theta(n-1) \quad \Rightarrow \quad T(n) = \Theta(n^2)$$

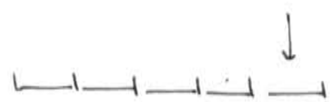
A more detailed pseudocode would be

```
Rec-Ins-Sort (A, 1, n)

    Rec-Ins-Sort (A, 1, n-1)

        key = A[n]
        i ← n-1
        while( i > 0 + A[i] > key) do
            do A[i+1] = A[i]
                i = i-1
            end do
        end do

End
```



with nth

1.3-5 Iterative Binary Search: input Array A sorted; key elt to search for, p + q elements of array to search from till to.

call Iterative Binary search IBS.

location = IBS(A, p, q, key)

```
location = nil
while( ~~p~~ q - p ≥ 0 ) do
    m = ⌊(p+q)/2⌋
    if (A[m] == key) then
        location = m
        ~~return~~ return
    elseif (A[m] > key) then
        q = m-1
    else
        p = m+1
    endif
enddo
```

Recursive Binary search RBS(A, p, q, key)

```
RBS(A, p, q, key)
    if (q - p < 0) return nil
    m = ⌊(p+q)/2⌋
```

```
if (A[m] == key)
    return m
elseif (A[m] > key)
    return RBS(A, p, m-1, key)
else
    return RBS(A, m+1, q, key)
endif
end
```

Worst case running time would be if we had to search $\log_2 n$ times.

Also from the recursive search

$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \log_2 n.$$

pg 15 CLR

(1.3-6) Insertion Sort (A) from pg #3

A[1, 2 ..., j-1] is sorted & we want to place A[j].

The answer is no. Even if we could "magically" obtain the location of placement of the elt A[j] we still must move all the elements down to insert A[j] into the array A[1, ..., j]. This ~~the~~ shifting of elements requires $\Theta(j)$ work. Thus the complexity of insertion sort is not changed from $\Theta(n^2)$. ~~but~~ Note in practice that this change would improve the performance of insertion sort

(1.3-7) Assume S has all positive #'s.

Sort the set S. complexity $\Theta(n \log_2 n)$ w/ merge sort.

place elt x in the list ~~so~~ of sorted S. Then the sum of two #'s can only come from elts that are smaller than x. This method will not work since if x is larger than every element. ~~the sorted list~~ I don't see how the sorted list would be of help.

Exhaustive search, requires

$$\binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2} \text{ additions.}$$

$$= \Theta(n^2)$$

PS 1? CLR

(1.4-1) $8n^2$ insert

$64 n \log_2 n$ merge

$$8n^2 < 64 n \log_2 n \Rightarrow 8n^2 - 64 n \log_2 n < 0$$

Solve $n^2 - 8n \log_2 n = 0$

$n - 8\log_2 n = 0 \Rightarrow n =$

$\log_2 n = \frac{\log_e n}{\log_e 2}$

n	$n - 8\log_2 n$
1	1
2	-6.0
3	-9.6
4	-12.0
5	-13.0
6	-14.0
10	-16.5
30	-9.2
40	-2.5
41	-1.8
42	-1.13
43	-.41
44	.324

Insersion sort Beats merge sort up to $n = 43$ from this size onward merge sort wins.

~~to~~ Rewrite merge sort to call insertion sort on inputs ~~less~~ less than or equal to 43.

pg 17 CLR

03-30-02 1

(1.4-2) $100n^2 < 2^n$

Find $n \ni$ $100n^2 - 2^n = 0$ $\Rightarrow n \cong 14$.

Thus for $n < 14$ Algo w/ running time $100n^2$ runs faster

Problem:

(1-1) $\log_2 n =$

$10^{+6} \log_2 n$ is time in seconds.

$10^{+6} \log_2 n = 1$ or can convert every time into μs.

$1s = 10^{+6} \mu s$

$1 min = 60 s = 6 \cdot 10^7 \mu s$

$1 hr = 3600 s = 3.6 \cdot 10^9 \mu s$

$1 day = 24(3.6) \cdot 10^9 = 8.64 \cdot 10^{10} \mu s$

$1 month = 2.592 \cdot 10^{12} \mu s$

$1 year = 3.1104 \cdot 10^{13} \mu s$

$1 century = 3.1104 \cdot 10^{15} \mu s$

$\cancel{\log_2 n = 10^{-6}}$

$\cancel{n = 2^{10^{-6}}} =$

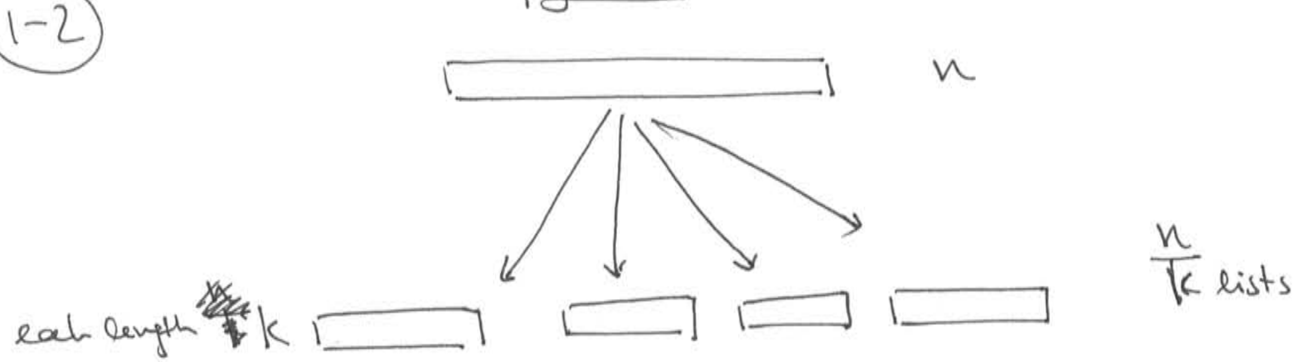
$\log_2 n = 10^{+6} \mu s = 1s$.

$n = 2^{10^{+6}}$

	10^{+6}	$6 \cdot 10^7$	$3,6 \cdot 10^9$	$8.64 \cdot 10^{10}$	$2.6 \cdot 10^{12}$	$3 \cdot 10^{13}$	$3 \cdot 10^{15}$
$\log_2 n$	$9 \cdot 10^{499}$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\sqrt{n}$	10^{12}	$3.6 \cdot 10^{15}$	$1.29 \cdot 10^{19}$	$7.46 \cdot 10^{21}$	$6.7 \cdot 10^{24}$	$9 \cdot 10^{26}$	$9 \cdot 10^{30}$
n	10^6	$6 \cdot 10^7$	$3.6 \cdot 10^9$	$8.64 \cdot 10^{10}$	$2.6 \cdot 10^{12}$	$3 \cdot 10^{13}$	$3 \cdot 10^{15}$
$n \log_2 n$							
n^2	10^3	$7.7 \cdot 10^3$	$6 \cdot 10^4$	$2.9 \cdot 10^5$	$1.6 \cdot 10^6$	$5.4 \cdot 10^6$	$5.47 \cdot 10^7$
n^3	100	391	1880	4400	13750	31,072	14425
2^n	19.	26					
$n!$	9	11	12	14	15	16	17.

Pg 17 CLR

(1-2)



(a) If each of the n/k lists are sorted w/ insertion sort. Then insertion sort requires $\Theta(k^2)$ time & to do $\frac{n}{k}$ of them would require

$$\frac{n}{k}\Theta(k^2) = \Theta(nk) \text{ time}$$

(b) To merge these $\frac{n}{k}$ lists into one list would require. I would have guessed $\Theta(n)$ work. Simply go along each of the sublists comparing the 1st element. & returning the minimum. a, b, c, d I don't see a way to do this w/o looping over a, b, c, d & thus touching $\frac{n}{k}$ elements. Thus I don't see any way to get $\log_2(\frac{n}{k})$

(c) $\Theta(nk + n\log_2(\frac{n}{k}))$ {merge sort $\Theta(n\log_2 n)$}

require that $\Theta(\underbrace{nk + n\log_2(\tfrac{n}{k})}) \sim \Theta(n\log_2 n)$

$$\Theta(nk - n\log_2 k + n\log_2 n)$$

Thus we require $\Theta(nk - n\log_2 k) \sim \Theta(n)$

or $\Theta(k - \log_2 k) \sim \Theta(1)$

$\cancel{k \approx \log_2 k}$ ~~for~~ $\cancel{k \approx}$

$k - \log_2 k \cong 1 \rightarrow k \cong 2$

$\cancel{k \log_2 k}$

(d) Based upon the value at which speed is optimum.

1-3

(a) $\langle 2, 3, 8, 6, 1 \rangle$

$(2,1), (3,1), (8,6), (8,1), (6,1)$

(b) $\langle n, n-1, n-2, \cdots, 2, 1 \rangle$

has $\cancel{n + n-1 + n-2 + \cdots + 2 + 1}$

$$n-1 + n-2 + \cdots + 2 + 1 = \sum_{k=1}^{n-1} k = \frac{n(n+1)}{2} - n$$

$$= \frac{n}{2}(n+1-2)$$

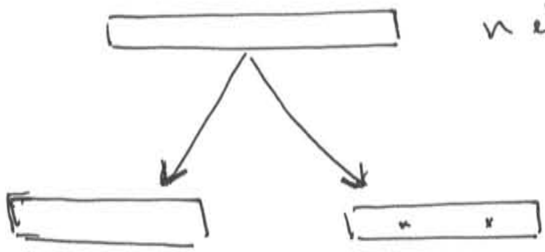
$$= \frac{n(n-1)}{2}$$

(c) The # of inversions of the sorted array w/ the unsorted additional element determines the # of moved elements there will be.

Thus. $\langle 1, 2, 3, 4, \overbrace{7, 8, 5}^{2} \rangle$ has 2 inversions

Thus / since the furthest inversion is 2 elts away 3 elts will have to be moved & the running time will be $\Theta(3)$. In fact. The # of inversions gives exactly the # of moved elements − 1.

(d)



n elts

n/2 elts each

$\langle 2, 3, 8, 6, 1 \rangle$ has 5 inversions

$\langle 2, 3, 8 \rangle$ 0 inversions

$\langle 6, 1 \rangle$ 1 inversion

$\langle 2, 3, 8, 6, 1 \rangle$

I claim one can get this information as we sort the list w/ merge sort. For example given

$\langle 2, 3, 8, 6, 1 \rangle$

$\langle 2, 3, 8 \rangle$ $\langle 6, 1 \rangle$

0 inversion $\langle 2, 3, 8 \rangle$ $\langle 1, 6 \rangle$ 1 inversion

A $\langle 2,3,8 \rangle$ B $\langle 1,6 \rangle$

count the # of times we take from the right list 1st

$\langle 1, 2, 3, 6, 8 \rangle$

2nd inv

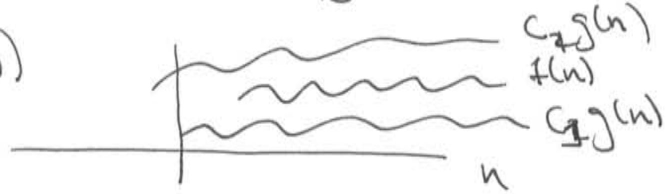
has (# elts in list A) inversions = 3

has # elts in list A inversions at this point = 1

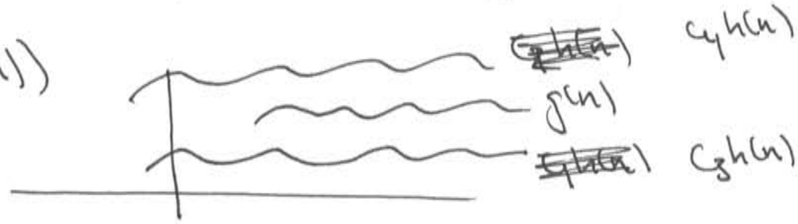
Thus the algorithm is the following. One performs merge sort. At every merging of two sorted arrays. The # of inversions required is none if we need all points from the left array. when we form the newly merged/sorted array each time we must pick an element from the right list the # of the # of inversions that go with this fact is the # of remaining elements in the left list. Thus as we form the merged list a running total of inversions is kept. Thus the time to compute the # of inversions is the same as sorting w/ merge sort or $\Theta(n \log_2 n)$

13 30 CLR

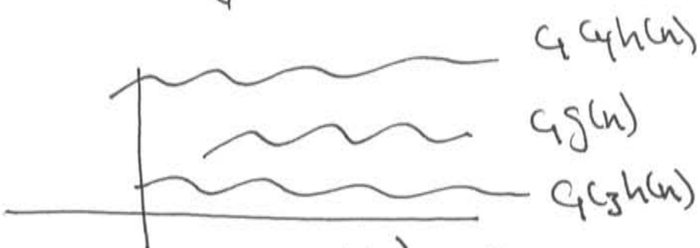
$f(n) = \Theta(g(n))$



$g(n) = \Theta(h(n))$

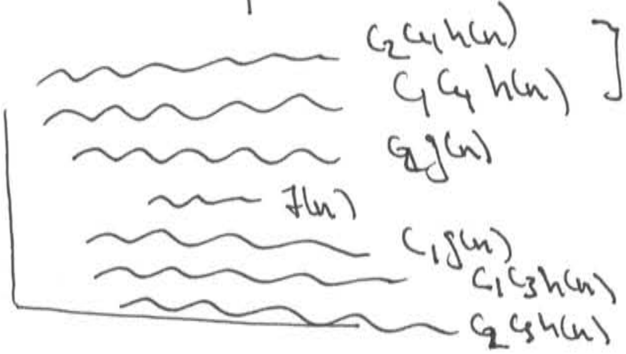


$\Downarrow$



+ sim. for $c_2 g(n)$

$\Rightarrow$



take max of $c_1 c_4$ + $c_2 c_4$

take max of $c_1 c_3$ + $c_2 c_3$

$\Rightarrow f(n) = \Theta(h(n))$

Pg 31 CLR

(2.1-1) $\max(f(n), g(n)) = \Theta(f(n) + g(n))$

$\iff \exists\ c_1 > 0,\ c_2 > 0\ \ n_0 \ni$

$0 \le c_1(f(n) + g(n)) \le \max(f(n), g(n)) \le c_2(f(n) + g(n))$

take $c_1 = 1/2$ Then

$\frac{1}{2}(f(n) + g(n)) \overset{?}{\le} \max(f(n), g(n))$

let $f(n) < g(n)$ Then $\frac{1}{2}(f(n) + g(n)) \overset{?}{\le} g(n)$

$$\frac{f(n)}{2} + \frac{g(n)}{2} \overset{?}{\le} g(n)$$

since $f(n) < g(n)$

$\frac{f(n)}{2} < \frac{g(n)}{2}$ $\therefore$ $\frac{f(n)}{2} + \frac{g(n)}{2} \le g(n)$ true

case where $f(n) > g(n)$ is symmetric.

take $c_2 = 1$

Then $\max(f(n), g(n)) \le f(n) + g(n)$ is true

$\therefore$ $\max(f(n), g(n)) = \Theta(f(n) + g(n))$

(2.1-2)

$$(n+a)^b = \Theta(n^b)$$

$$\parallel$$

$$n^b\left(1+\frac{a}{n}\right)^b \le (1+a)^b n^b \qquad \text{since } \frac{a}{n} < a \quad \forall n \ge 1$$

Thus $c_2 = (1+a)^b$ works. Also $\quad 1 \le 1 + \frac{a}{n} \quad \forall n \ge 1$

Thus $c_1 = 1$ works

$$n^b \le n^b\left(1+\frac{a}{n}\right)^b \le (1+a)^b n^b \qquad \forall n \ge 1.$$

(2.1-6) $\quad T(n) = \Theta(g(n)) \iff T_w(n) = O(g(n))$ +

$T_B(n) = \Omega(g(n))$

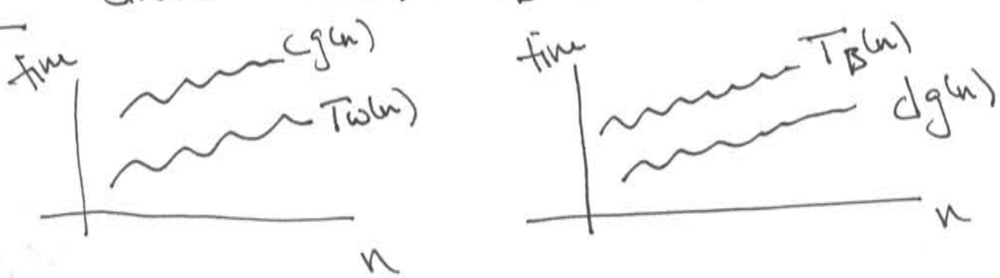
$\Rightarrow$ No matter what inputs $T(n) = \Theta(g(n))$ (see pg 28)

Thus take Best + worst inputs.

Th $\quad T_w(n) = \Theta(g(n))$ + $T_B(n) = \Theta(g(n))$

$\Rightarrow T_w(n) = O(g(n))$ + $T_B(n) = \Omega(g(n))$

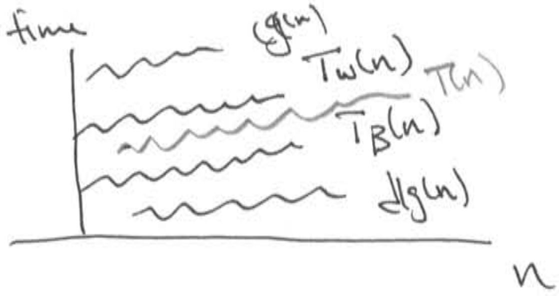
$\Leftarrow$ since $T_w(n) > T_B(n)$ By definition. Plotting the relationships



$T_w(n) = O(g(n))$

$T_B(n) = \Omega(g(n))$ gives

the plots to the left

$\Rightarrow$ 

since T(n) time for arbitrary input $\ni$ $T_W(n) > T(n) > T_B(n)$

The ~~pict~~ picture above shows $T(n) = \Theta(g(n))$.

(2.1-7) Pr: $o(g(n)) \cap \omega(g(n)) = \phi$

let $f(n) \in o(g(n)) \cap \omega(g(n))$ then as $f(n) \in o(g(n))$

$\Rightarrow \forall$ any $c > 0$ $\exists$ $n_0 > 0$ $\ni$ $f(n) < c\,g(n)$ $\forall n > n_0$.

as $f(n) \in \omega(g(n))$

$\Rightarrow \forall$ any $c > 0$ $\exists$ $n_0 > 0$ $\ni$ $f(n) > c\,g(n)$ $\forall n > n_0$.

But choosing ~~the~~ a c & a n_0 from the def of $f(n) \in o(g(n))$ would give a contradiction to the def of $f(n) \in \omega(g(n))$, in contradiction to our assumption that ~~set~~ ~~other~~ $o(g(n)) \cap \omega(g(n)) \neq \phi$.

(2.1-8) $\Omega(g(n,m)) = \{ f(n,m)$: $\exists$ positive constants c_1 & n_0, m_0 $\ni$ $0 \le c_1 g(n,m) \le f(n,m)$ $\forall$ $n \ge n_0$ & $m \ge m_0 \}$

$\Theta(g(n,m)) = \{ f(n,m)$: $\exists$ positive constants $c_1, c_2, n_0,$ & m_0 $\ni$ $0 \le c_1 g(n,m) \le f(n,m) \le c_2 g(n,m)$ $\forall$ $n \ge n_0$ & $m \ge m_0 . \}$

pg 32 CLR

$$x-1 \le \lfloor x \rfloor \le x \le \lceil x \rceil \le x+1$$

$$\left\lceil \lceil n/a \rceil \frac{1}{b} \right\rceil = \left\lceil \frac{n}{ab} \right\rceil \qquad n, a, b \text{ integers}$$

$$\left\lfloor \lfloor n/a \rfloor \frac{1}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor$$

Pr:

pg 34 CLR

$$\log_b a = \frac{1}{\log_a b}$$

$$\| $$

$$\frac{\ln a}{\ln b} = \frac{1}{\frac{\ln b}{\ln a}} = \frac{1}{\log_a b}$$

eq 2.5 $$\lim_{n \to +\infty} \frac{n^b}{a^n} = 0$$

$$\lim_{n \to +\infty} \frac{(\log_2 n)^b}{(2^a)^{\log_2 n}} = \lim_{n \to +\infty} \frac{(\log_2 n)^b}{2^{a \log_2 n}} = \lim_{n \to +\infty} \frac{(\log_2 n)^b}{n^a} = 0$$

$$\Rightarrow \quad \log_2 n^b = o(n^a)$$

— — ———

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right) = n^n \sqrt{2\pi} \frac{\sqrt{n}}{e^n}(1 + \Theta(1/n))$$

Now $$\lim_{n \to +\infty} \frac{n!}{n^n} = \lim_{n \to +\infty} \frac{\sqrt{n}}{e^n}(1 + \Theta(1/n)) = 0$$

$$\Rightarrow \quad n! = o(n^n)$$

As $f(n) = \omega(g(n)) \Rightarrow \lim_{n \to +\infty} \frac{f(n)}{g(n)} = +\infty$

Thus $$\lim_{n \to +\infty} \frac{n!}{2^n} = \lim_{n \to +\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \Theta(1/n))}{2^n}$$

~~Pg 34 CLR~~

$$\lim_{n \to +\infty} \frac{\sqrt{2\pi n}\, n^n (1+\Theta(1/n))}{(2e)^n} = +\infty$$

$$\log_2 n! = \log_2\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1+\Theta(1/n))\right]$$

$$= \frac{1}{2}\log_2 2\pi + \frac{1}{2}\log_2 n + n\log_2 n - n\log_2 e + \log_2(1+\Theta(1/n)).$$

$$= \frac{1}{2}\log_2 2\pi + \log_2(1+\Theta(1/n)) + \frac{1}{2}\log_2 n - n\log_2 e + n\log_2 n$$

Now

$$\log_2 n! = \Theta(n\log_2 n) \iff \lim_{n\to+\infty} \frac{\log_2 n!}{n\log_2 n} = c$$

$$= \lim_{n\to+\infty}\left(\frac{\frac{1}{2}\log_2 n}{n\log_2 n} - \frac{n\log_2 e}{n\log_2 n} + \frac{n\log_2 n}{n\log_2 n}\right) = 1$$

$$\iff \cancel{\exists\ n_0 > 0 \ni}\ \forall \epsilon > 0\ \exists\ n_0 \ni \quad \forall n \geq n_0$$

$$\left| \frac{\log_2 n!}{n\log_2 n} - 1 \right| < \epsilon$$

~~Thus pick $\epsilon = 2$. Then $\exists$ $n_0 > 0 \ni$~~

$$\cancel{< \frac{\log_2 n!}{n\log_2 n} \ni \Leftarrow 2}$$

$\iff$

$$1-\epsilon \leq \frac{\log_2 n!}{n\log_2 n} \leq \epsilon+1 \qquad \forall n \geq n_0$$

$\Leftrightarrow \quad (1-\epsilon)\, n\log_2 n \le \log_2 n! \le (\epsilon+1)\, n\log_2 n$

let $\epsilon = 1/2$ Then

$$\frac{1}{2} n\log_2 n \le \log_2 n! \le \frac{3}{2} n\log_2 n$$

pg 37 CLR

2.2-2

$$T(n) = n^{O(1)} \Rightarrow T(n) \in n^{k(n)} \ni k(n) = O(1) \Rightarrow k(n) \leq c_1$$

$$\Rightarrow k(n) = k \Rightarrow T(n) \in O(n^k).$$

If $T(n) \in O(n^k)$

$$\Rightarrow T(n) \leq cn^k \quad \forall n \geq n_0.$$

$$\leq cn^{O(1)} \Rightarrow T(n) = n^{O(1)}$$

2.2-3 Pr: $a^{\log_b n} = n^{\log_b a}$

let $y = a^{\log_b n} \Rightarrow \log_b y = (\log_b n)(\log_b a)$

$\cancel{\ln y = \log_b n \ln a}$

$$= (\log_b a) \cdot (\log_b n)$$

$$= \log_b n^{\log_b a}$$

$$\Rightarrow y = n^{\log_b a} = a^{\log_b n}$$

2.2-4 Pr $\log_2 n! = \Theta(n \log_2 n)$ see notes w/ pg 35.

$n! = o(n^n)$ see notes pg 35

pg 37 CLR

04-01-02 1

(2.2-5) $\lceil \log_2 n \rceil! \stackrel{?}{=} O(n^k)$ some constant k.

let $n = 2^k$ $k = 1, 2, 3, \ldots$

Then $\lceil \log_2 2^k \rceil! = k! = (\log_2 n)!$

$= k^k$

(2.2-6) $\log_2(\log^* n)$ or $\log^*(\log_2 n)$

let $n = 2^{65536}$

$\log^* n = 5$ $\log_2 n = 65536$

$\log_2(\log^* n) = 2.32$ $\log^*(\log_2 n) = 5$. Thus by inspection it ~~wo~~ would appear that $\log^*(\log_2 n)$ is larger. This can be seen w/ the following argument.

$\log^* n = \log^*(\log_2 n) + 1$ for ~~thing~~ large n since

The $\log_2 n$ will take away from the # of $\log_2 n$ applications

That need to be applied. since $\log^*(\log_2 n)$

My guess is $\log^*(\log_2(n))$ Bounds $\log_2(\log^* n)$ assymptotically

But I'm not sure how to show it.

04-01-021

Pg 37 CLR

(2.2-7)

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

$F_0 = 0$ ✓ $\quad F_1 = \dfrac{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} = 1$ ✓

Assume $F_i = \dfrac{\phi^i - \hat{\phi}^i}{\sqrt{5}} \quad \forall\, i \le n$

Pr true for F_{n+1}

$$F_{n+1} = F_n + F_{n-1}$$

$$= \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} + \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}}$$

$$= \frac{\phi^n + \phi^{n-1}}{\sqrt{5}} - \left(\frac{\hat{\phi}^n + \hat{\phi}^{n-1}}{\sqrt{5}}\right) =$$

$$= \phi^{n+1}\left(\frac{\phi^{-1} + \phi^{-2}}{\sqrt{5}}\right) - \hat{\phi}^{n+1}\left(\frac{\hat{\phi}^{-1} + \hat{\phi}^{-2}}{\sqrt{5}}\right)$$

<u>Now</u>: $\quad \phi^{-1} + \phi^{-2} = \dfrac{2}{1+\sqrt{5}} + \dfrac{4}{(1+\sqrt{5})^2} = \dfrac{2(1+\sqrt{5}) + 4}{(1+\sqrt{5})^2}$

$$= \frac{6 + 2\sqrt{5}}{(1+\sqrt{5})^2}\,\frac{(1-\sqrt{5})^2}{(1-\sqrt{5})^2} = \frac{(6+2\sqrt{5})(1 - 2\sqrt{5} + 5)}{(1-5)^2}$$

$$\phi^{-1} + \phi^{-2} = \frac{(6+2\sqrt{5})(6-2\sqrt{5})}{16} = \frac{36-4\cdot 5}{16} = 1$$

& $$\hat{\phi}^{-1} + \hat{\phi}^{-2} = \frac{2}{1-\sqrt{5}} + \frac{4}{(1-\sqrt{5})^2} = \frac{2(1-\sqrt{5})+4}{(1-\sqrt{5})^2}$$

$$= \frac{(6-2\sqrt{5})}{(1-\sqrt{5})^2}\frac{(1+\sqrt{5})^2}{(1+\sqrt{5})^2} = \frac{(6-2\sqrt{5})(1+2\sqrt{5}+5)}{(1-5)^2}$$

$$= \frac{(6-2\sqrt{5})(6+2\sqrt{5})}{16} = \frac{36-4\cdot 5}{16} = 1$$

Thus

$$F_{n+1} = \frac{\phi^{n+1}}{\sqrt{5}} - \frac{\hat{\phi}^{n+1}}{\sqrt{5}}$$

pg 37 CLR

04-01-02

(2.2-8) Pr:
$F_{i+2} \geq \phi^i$

$$F_{i+2} = \frac{\phi^{i+2} - \hat{\phi}^{i+2}}{\sqrt{5}} = \phi^i \left[\frac{\phi^2 - \hat{\phi}^{i+2} \phi^{-i}}{\sqrt{5}} \right]$$

$$= \phi^i \left[\frac{\phi^2 - \hat{\phi}^i \phi^i \hat{\phi}^2}{\sqrt{5}} \right] = \phi^i \left[\frac{\phi^2 - (\phi\hat{\phi})^i \hat{\phi}^2}{\sqrt{5}} \right]$$

Now $\phi\hat{\phi} = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1-5}{4} = -1$

Thus
$$F_{i+2} = \phi^i \left[\frac{\phi^2 - (-1)^i \hat{\phi}^2}{\sqrt{5}} \right]$$

Now consider

$$\frac{\phi^2 + \hat{\phi}^2}{\sqrt{5}} \quad + \quad \frac{\phi^2 - \hat{\phi}^2}{\sqrt{5}}$$

$$\phi^2 + \hat{\phi}^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 + \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1+\cancel{2\sqrt{5}}+5}{4} + \frac{1-\cancel{2\sqrt{5}}+5}{4} = \frac{12}{4} = 3$$

$$\phi^2 - \hat{\phi}^2 = \quad = \frac{\cancel{1}+2\sqrt{5}+\cancel{5}}{4} - \frac{\cancel{1}-2\sqrt{5}+\cancel{5}}{4} = \frac{4\sqrt{5}}{4} = \sqrt{5}$$

Thus $F_{i+2} = \phi^i \left[\frac{\phi^2 - (-1)^i \hat{\phi}^2}{\sqrt{5}} \right] \geq \phi^i \left[\frac{\phi^2 - \hat{\phi}^2}{\sqrt{5}} \right] = \phi^i$

pg 38 CLR

(2-1) $p(n) = \sum_{i=0}^{d} a_i n^i$

(a) ~~[illegible]~~

If $k \geq d$, then $p(n) = O(n^k)$

$\iff$ ~~$p(n)$~~ $\exists$ constants c + $n_0 > 0$ $\ni$

$0 \leq p(n) \leq cn^k \quad \forall\ n \geq n_0$

?

$\iff$ $\exists$ constants c + $n_0 > 0$ $\ni$

$$0 \leq \sum_{i=0}^{d} a_i n^i \leq cn^k \quad \forall\ n \geq n_0.$$

$$\iff 0 \leq \sum_{i=0}^{d} a_i n^{i-k} \leq c \quad \forall\ n \geq n_0$$

$$\iff 0 \leq a_0 n^{-k} + a_1 n^{1-k} + \cdots + a_d n^{d-k} \leq c \quad \forall n \geq n_0$$

Since $\lim_{n \to +\infty} a_0 n^{-k} + a_1 n^{1-k} + \cdots + a_d n^{d-k} = 0$ the above statement is true.

(b) If $k \leq d$ then $p(n) = \Omega(n^k)$

$\iff$ $\exists$ constants c + $n_0 > 0$ $\ni$

$0 \leq cg(n) \leq f(n) \quad \forall\ n \geq n_0$

$\iff 0 \leq cn^k \leq p(n) \quad \forall\ n \geq n_0$

$$\iff 0 \leq c \leq \sum_{i=0}^{d} a_i n^{i-k} \quad \forall\ n \geq n_0$$

$\iff \quad 0 \le c \le a_0 n^{-k} + a_1 n^{1-k} + \cdots + a_d n^{d-k} \quad \forall\, n \ge n_0$

Since $k < d$ & $\lim\limits_{n\to+\infty} a_0 n^{-k} + a_1 n^{1-k} + \cdots + a_d n^{d-k} = +\infty$

This is true.

(c) If $k = d$ then $p(n) = \Theta(n^k)$

$\iff \exists\, c_1, c_2 \,\&\, n_0 > 0 \ni$

$0 \le c_1 n^k \le p(n) \le c_2 n^k \quad \forall\, n \ge n_0.$

$\iff$

$$c_1 \le a_0 n^{-k} + a_1 n^{1-k} + a_2 n^{2-k} + \cdots + a_d n^{d-k} \le c_2 \quad \forall\, n \ge n_0$$

$$c_1 \le a_0 n^{-k} + a_1 n^{1-1} + \cdots + a_d \le c_2 \quad \forall\, n \ge n_0.$$

Since $a_0 n^{-k} + a_1 n^{1-k} + \cdots + a_{d-1} n^{-1} \longrightarrow 0$ as $n \to +\infty$

The above statement is true

(d) If $k > d$ then $p(n) = o(n^k)$

$\iff \lim\limits_{n\to+\infty} \dfrac{p(n)}{n^k} = 0$ which is true.

(e) If $k < d$ then $p(n) = \omega(n^k)$

$\iff \lim\limits_{n\to+\infty} \dfrac{p(n)}{n^k} = +\infty$ which is true.

PS3 CLR

(2-2) $A = \square(B) \quad \forall\ \square = O, o, \Omega, \omega, \Theta$

(a)

$$\log_2^k n = O(n^\epsilon) \Rightarrow \log_2^k n \le Cn^\epsilon \quad \forall n \ge n_0$$

$$\Leftrightarrow \frac{\log_2^k n}{n^\epsilon} \le C$$

$$\lim_{n\to+\infty} \frac{\log_2^k n}{n^\epsilon} \overset{\infty/\infty}{=} \frac{k\log_2^{k-1} n\,(1/n)\,c_1}{\epsilon n^{\epsilon-1}} = c_1 k \frac{\log_2^{k-1} n}{\epsilon n^{\epsilon}} = 0$$

We see that $\log_2^k n = O(n^\epsilon)$

Also $\log_2^k n = o(n^\epsilon)$

Is $\log_2^k n = \Omega(n^\epsilon) \Leftrightarrow \exists\ c + n_0 > 0 \ni$

$$0 \le cn^\epsilon < \log_2^k n \quad \forall n \ge n_0$$

Which is false

(b) $n^k = \square(c^n)$ one can show

$n^k = O(c^n)$ + $n^k = o(c^n)$ but no other relationships hold

(c) $\sqrt{n}$ + $n^{\sin n}$ cannot be compared for large n since the exponent on $n^{\sin n}$ takes values between ± 1.

(d) $2^n = 2^{n/2}$ one can show

$$2^n = \Omega(2^{n/2}) \quad + \quad 2^n = \omega(2^{n/2})$$

Check $\overset{?}{\exists}\ c + n_0 > 0 \ni$

$$c2^{n/2} \le 2^n \quad \forall\ n \ge n_0$$

$$c \le 2^{n/2} \quad \forall\ n \ge n_0 \quad \text{yes pick } c = 2^{1/2}.$$

(e) $n^{\log_2 m}$ $\quad m^{\log_2 n}$

$$n^{\log_2 m} = O(m^{\log_2 n}) \quad + \quad n^{\log_2 m} = o(m^{\log_2 n})$$

(f) Stirlings formula gives us $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta(1/n)\right)$

since from pg 35

$$\log_2 n! = \Theta(n \log_2 n)$$

$+ \log_2 n^n = n \log_2 n$ one has that

$\log_2 n! = O(\log_2 n^n)$; $\log_2 n! = \Omega(\log_2 n^n)$; $\log_2 n! = \Theta(\log_2 n^n)$

The chart would thus look like the following:

A	B	O	o	Ω	ω	Θ
$\lg^k n$	n^t	Yes	Yes	No	No	No
n^k	c^n	Yes	Yes	No	No	No
$\sqrt{n}$	$n^{\sin n}$	No	No	No	No	No
2^n	$2^{n/2}$	No	No	Yes	Yes	No
$n^{\log_2 m}$	$m^{\log_2 n}$	Yes	Yes	No	No	No
$\log_2(n!)$	$\log_2(n^n)$	Yes	No	Yes	No	Yes

pg 38 CLR

2-3 $g_1, g_2, g_3, \dots, g_{30}$

$g_1 = \Omega(g_2) \Rightarrow \exists\ c + n_0 > 0 \ni$

$0 \le c g_2 \le g_1 \quad \forall\ n \ge n_0$

Thus g_1 is the "largest" fn + g_{30} is the "smallest" fn.

pg 39 CLR

2-4

(a) False $n = O(n^2)$ But $n^2 \neq O(n)$

(b) False let $f(n) = \frac{1}{n}$ $g(n) = n$

Then $\frac{1}{n} + n \stackrel{?}{=} \Theta(\min(\frac{1}{n}, n)) = \Theta(1/n)$ is false.

(c) ~~F~~ True $f(n) = O(g(n))$

$\Rightarrow \exists\ c > 0, \ \&\ n_0 > 0 \ \ni$

$0 \le \cancel{g(n)}\ f(n) \le c g(n) \quad \forall\ n \geq n_0.$

$\Rightarrow \quad \log_2 f(n) \le \log_2 g(n) + \log_2 c \quad \forall\ n \geq n_0$

$\Rightarrow \quad \log_2 f(n) \le 2 \log_2 g(n)$

$\Rightarrow \quad \log_2 f(n) = O(\log_2 g(n)).$

(d) ~~True~~ False $f(n) = O(g(n))$

$\Rightarrow \exists\ c\ \&\ n_0 > 0 \ \ni$

$0 \le f(n) \le c g(n) \quad \forall\ n \geq n_0$

$\Rightarrow \quad 2^{f(n)} \le 2^{c g(n)}$ "

$(2^c)^{g(n)}$

If say $c = 2$ $\quad 2^{f(n)} \le 4^{g(n)}$ & $2^{f(n)} = O(4^{g(n)})$ but not $O(2^{g(n)})$

(e) False let $f(n) = \frac{1}{n}$

Then $f(n) \neq O(\frac{1}{n^2})$ since $f(n) > \frac{1}{n^2}$

(f) True $f(n) = O(g(n)) \Rightarrow \exists\ c \ \& \ n_0 > 0 \ \ni$

$0 \leq f(n) \leq c g(n) \quad \forall \ n \geq n_0.$

$\Leftrightarrow \quad 0 \leq \frac{1}{c} f(n) \leq g(n) \quad \forall \ n \geq n_0.$

$\Rightarrow \exists \ c_2 \ \& \ n_0 \ \ni$

$0 \leq c_2 f(n) \leq g(n) \quad \forall \ n \geq n_0$

$\Rightarrow \quad g(n) = \Omega(f(n))$

(g) ~~False~~: $\overset{?}{\exists}\ c_1 \ \& \ c_2 \ \& \ n_0 > 0 \ \ni$
False.

$0 \leq c_1 f(n/2) \leq f(n) \leq c_2 f(n/2) \quad \forall \ n \geq n_0$

False take $f(n) = n^n$ Then ~~or~~

$0 \leq c_1 \left(\frac{n}{2}\right)^{n/2} \leq n^n \leq c_2 \left(\frac{n}{2}\right)^{n/2}$ cannot be true

or $f(n) = n!$

$0 \leq c_1 (\frac{n}{2})! \leq n! \leq c_2 (\frac{n}{2})!$ False. $\forall \ n \geq n_0$

(h) $f(n) + o(f(n)) = \Theta(f(n))$

True. $f(n) + o(f(n)) = f(n) + g(n)$ ∍ $g(n) = o(f(n))$

$\{ \forall c>0 \ \exists \text{ a } n_0>0 \quad 0 \le f(n) \le c\,g(n) \ \forall n \ge n_0 \}$

$\Rightarrow \quad \forall c>0 \ \exists \text{ a } n_0>0 \quad 0 \le g(n) \le c f(n) \ \forall n \ge n_0.$

Thus $\overset{?}{\exists}\ c_1 + c_2 > 0$ & $n_0 > 0$ ∍

$$0 \le c_1 f(n) \le f(n) + g(n) \le c_2 f(n) \quad \forall n \ge n_0.$$

pick $c_1 = 1$ & $c_2 = 2$ & pick c in the definition of $g(n) = o(f(n))$ equal 1. Then $0 \le g(n) \le 1 f(n) \ \forall n \ge n_0$

$$\therefore \quad 0 \le f(n) \underset{\substack{\uparrow \\ \text{yes since} \\ g(n)>0}}{\overset{?}{\le}} f(n) + g(n) \le f(n) + f(n) = 2f(n) \underset{\substack{\uparrow \\ \text{yes}}}{\le} 2f(n)$$

$\therefore \quad f(n) + o(f(n)) = \Theta(f(n))$

pg 39 CLR 04-01-02 1

(2-5)

(a) ~~Ex~~ ~~Ex~~ $\cancel{f(n) = O(g(n))}$

~~consider f(n) - g(n). Then ∀ n value either f(n) - g(n) > 0, = 0, < 0~~

~~If f(n) - g(n) < 0 for all every n then f(n) = O(g(n))~~

~~Ex~~ f & g are asymptotically non negative $\Rightarrow \exists\, n_1 > 0 \ni$

$f(n) > 0 \quad \forall \quad n \geq n_1$

& $\exists\, n_2 > 0 \ni \quad g(n) > 0 \quad \forall\, n \geq n_2.$

04-01-02 1

pg 45 CLR

(3.1-1) $$\sum_{k=1}^{n} 2k-1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= 2\left[\frac{n(n+1)}{2} - 0\right] - n = n(n+1) - n = n^2$$

(3.1-2) Given that $\sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$

n even say 4 $\lfloor n/2 \rfloor = 2$
1 2 3 4
n odd say 5
1 2 3 4 5 $\lfloor n/2 \rfloor = 2$
$\lceil n/2 \rceil = 3$

Assume n is even.

$$\underbrace{\sum_{k=1}^{n/2} \frac{1}{2k}}_{\text{even terms}} + \underbrace{\sum_{k=1}^{n/2} \frac{1}{2k-1}}_{\text{odd terms}} = \ln n + O(1)$$

$$\frac{1}{2}\sum_{k=1}^{n/2} \frac{1}{k} + \sum_{k=1}^{n/2} \frac{1}{2k-1} = \ln n + O(1)$$

$$\|$$

$$\frac{1}{2}\ln(n/2) + O(1) + \sum_{k=1}^{n/2} \frac{1}{2k-1} = \ln n + O(1)$$

$$\Rightarrow \sum_{k=1}^{n/2} \frac{1}{2k-1} = \ln n - \ln(\sqrt{n}) + \ln(\sqrt{2}) + O(1)$$

$$= \ln(\sqrt{n}) + O(1)$$

Same expression holds when n is odd

RBZ

pg 45 CLR

3.1-3 $$\sum_{k=0}^{\infty} \frac{k-1}{2^k} = 0$$

$$\parallel$$

$$\sum_{k=0}^{\infty} k2^{-k} - \sum_{k=0}^{+\infty} \left(\tfrac{1}{2}\right)^k = 0$$

$$\parallel$$

$$\frac{1/2}{(1-1/2)^2} - \frac{1}{1-(1/2)} = \frac{1/2}{1/4} - \frac{1}{1/2}$$

By eq 3.6

$$= 2 - 2 = 0 \quad \checkmark$$

3.1-4 $$\sum_{k=1}^{\infty} (2k+1)x^{2k} = f(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^{2k+1}\right)$$

$$\cancel{xf(x)} = \cancel{\sum_{k=1}^{\infty}(2k+1)x^{2k+1}} \neq \cancel{\frac{d}{dx}} = \frac{d}{dx}\left(x\sum_{k=0}^{\infty}(x^2)^k\right)$$

$$\therefore f(x) = \frac{d}{dx}\left(x \frac{1}{1-x^2}\right) = \frac{d}{dx}\left(\frac{x}{1-x^2}\right)$$

pg 45 CLR

(3.1-5) $$\sum_{k=1}^{m} O(f_k(n)) \Rightarrow \sum_{k=1}^{m} g_k(n) \quad \ni \quad g_k(n) = O(f_k(n)) \ \forall k = 1, \dots, m$$

Then $\Rightarrow$ $g_k(n) \le c_k f_k(n)$

Summing for $k = 1, \dots, m$ gives

$$\sum_{k=1}^{m} g_k(n) \le \sum_{k=1}^{m} c_k f_k(n) \le \max_k c_k \sum_{k=1}^{m} f_k(n) = C \sum_{k=1}^{m} f_k(n)$$

$$\Rightarrow \sum_{k=1}^{m} O(f_k(n)) = O\left(\sum_{k=1}^{m} f_k(n)\right)$$

(3.1-6) Pr: $$\sum_{k=1}^{\infty} \Omega(f_k(n)) = \Omega\left(\sum_{k=1}^{\infty} f_k(n)\right)$$

L.H.S $= \sum_{k=1}^{\infty} g_k(n) \quad \ni \quad g_k(n) = \Omega(f_k(n))$

$\Rightarrow c_k f_k(n) \le g_k(n) \quad \forall \ n \ge n_k$

$$\min_k(c_k) \sum_{k=1}^{\infty} f_k(n) \le \sum_{k=1}^{\infty} c_k f_k(n) \le \sum_{k=1}^{\infty} g_k(n)$$

$$\Rightarrow \sum_{k=1} \Omega(f_k(n)) = \Omega\left(\sum_{k=1} f_k(n)\right)$$

pg 46 CLR

3.1-7 $\prod_{k=1}^{n} 2\cdot 4^k = 2\prod_{k=1}^{n} 2^{2k} = 2(2^2\cdot 2^4\cdot 2^6\cdots 2^{2n})$

$= 2^{1+2+4+6+\cdots+2n} = 2^{1+2(1+2+3+\cdots+n)} = 2^{1+2\left(\frac{n(n+1)}{2}\right)} = 2^{1+n(n+1)}$

$= 2^{n^2+n+1}$

3.1-8 $\prod_{k=2}^{n}(1-1/k^2) = \prod_{k=2}^{n}\left(\frac{k^2-1}{k^2}\right) = \prod_{k=2}^{n}\frac{(k-1)(k+1)}{k^2}$

$= \prod_{k=2}^{n}\left(\frac{k-1}{k}\right)\left(\frac{k+1}{k}\right)$

$= \left(\frac{1}{2}\cdot\frac{\cancel{3}}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{4}}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{5}}{\cancel{4}}\right)\cdots\left(\frac{\cancel{n-1}}{\cancel{n}}\cdot\frac{n+1}{n}\right) = \frac{1}{2}\left(\frac{n+1}{n}\right) = \frac{1}{2}\left(1+\frac{1}{n}\right)$

pg 50 CLR

$$\sum_{k=1}^{n} \frac{1}{k}$$

$$n = 2^{\lfloor \lg n \rfloor} + 2$$

$$\sum_{j=0}^{0} \frac{1}{1+j} + \sum_{j=0}^{1} \frac{1}{2+j} + \sum_{j=0}^{3} \frac{1}{4+j} + \cdots + \sum_{j=0}^{2^{\lfloor \log_2 n \rfloor}-1} \frac{1}{2^{\lfloor \log_2 n \rfloor} + j}$$

$$= \underbrace{\frac{1}{1}} + \underbrace{\frac{1}{2} + \frac{1}{3}} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}} + \underbrace{\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}}$$

$$+ \cdots + \frac{1}{2^{\lfloor \log_2 n \rfloor}} + \frac{1}{2^{\lfloor \log_2 n \rfloor} + 1} + \frac{1}{2^{\lfloor \log_2 n \rfloor} + 2} + \cdots + \frac{1}{2^{\lfloor \log_2 n \rfloor} + 2^{\lfloor \log_2 n \rfloor} - 1}$$

$$= \underbrace{\frac{1}{1}} + \underbrace{\frac{1}{2} + \frac{1}{3}} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots +$$

$$\frac{1}{2^{\lfloor \log_2 n \rfloor}} + \frac{1}{2^{\lfloor \log_2 n \rfloor} + 1} + \frac{1}{2^{\lfloor \log_2 n \rfloor} + 2} + \cdots + \frac{1}{2^{\lfloor \log_2 n \rfloor + 1} - 1}$$

Thus $\log_2 n$ would be the exact # of powers of two inside

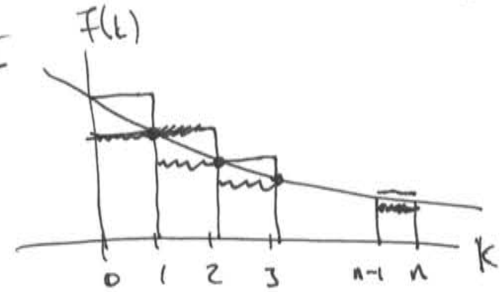
But this # may be fractional

$$\lfloor \log_2 n \rfloor \le \log_2 n \le \lfloor \log_2 n \rfloor + 1$$

Thus we see that we are adding additional $\frac{1}{2^p + j}$ elements beyond what the original sum called for.

Pg 50 CLR

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$



$$\int_1^{n+1} \frac{dx}{x} \leq \sum_{k=1}^{n} \frac{1}{k} \leq \int_0^{n} \frac{dx}{x}$$

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_1^{n} \frac{dx}{x} = \ln n$$

$$\therefore \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1$$

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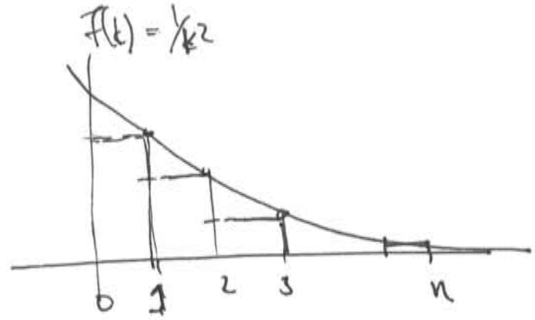
Pg 52 CLR

(3-2.1) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \le \int_0^n \frac{dx}{x^2} =$



$$= \int_0^n x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_0^n$$

$$= -\left(\frac{1}{n} - \infty\right) = +\infty .$$

$$\sum_{k=2}^{n} \frac{1}{k^2} \le \int_1^n \frac{dx}{x^2} = \left.-\frac{1}{x}\right|_1^n = -\left(\frac{1}{n} - 1\right) = 1 - \frac{1}{n}$$

$$\therefore \sum_{k=1}^{n} \frac{1}{k^2} \le 2 - \frac{1}{n} < 2$$

(3.2-2) $\displaystyle\sum_{k=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^k} \right\rceil \le \sum_{k=0}^{\lfloor \lg n \rfloor} \left(\frac{n}{2^k} + 1\right) \le \lfloor \lg n \rfloor + 1 + \sum_{k=0}^{\lfloor \lg n \rfloor} \frac{n}{2^k}$

$$\le \lfloor \lg n \rfloor + 1 + n \sum_{k=0}^{\lfloor \lg n \rfloor} \frac{1}{2^k}$$

$$\leq \lfloor \log_2 n \rfloor + 1 + n\left(\frac{1-(2^{-1})^{\lfloor \lg n \rfloor + 1}}{1-2^{-1}}\right) \qquad \sum_{k=0}^{n_2} r^k = \frac{1-r^{n_2+1}}{1-r}$$

$$\sum_{k=n_1}^{n_2} \neq \frac{}{}$$

$$= \lfloor \log_2 n \rfloor + 1 + 2n$$

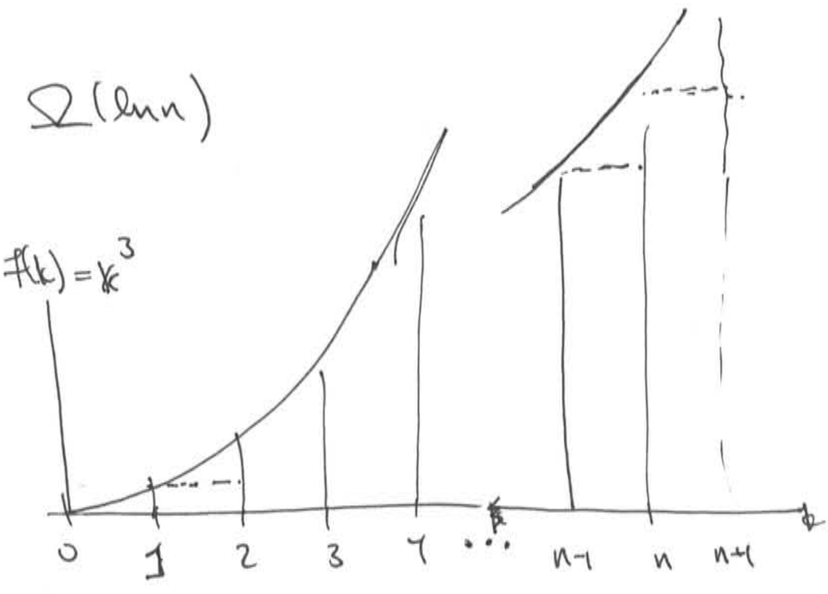
3.2-3 $\quad H_n = \sum_{k=1}^{n} \frac{1}{k} = \underset{\text{even terms}}{\sum} + \underset{\text{odd terms}}{\sum} \quad ?$

From 3.11 $\quad \sum_{k=1}^{n} \frac{1}{k} \geq \ln(n+1) \neq \cancel{\Omega(\ln n)}$

$$= \ln n + \ln(1+\tfrac{1}{n}) \geq \ln n$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k} = H_n = \Omega(\ln n)$$

3.2-4 $\quad \sum_{k=1}^{n} k^3$



$$\int_0^n x^3\,dx \leq \sum_{k=1}^{n} k^3 \leq \int_1^{n+1} x^3\,dx$$

$$\frac{n^4}{4} = \frac{x^4}{4}\Big|_0^n \leq \sum_{k=1}^{n} k^3 \leq \frac{x^4}{4}\Big|_1^{n+1} = \frac{(n+1)^4}{4} - \frac{1}{4}$$

(3.2-4)

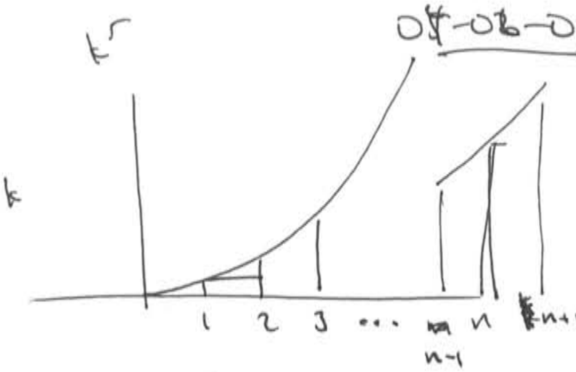
$$\frac{n^4}{4} \le \sum_{k=1}^{n} k^3 \le \frac{1}{4}\left((n+1)^4 - 1\right)$$

(3.2-5) Singular pt at 0.

Pg 52 CLR 04-06-021

Prob 3-1

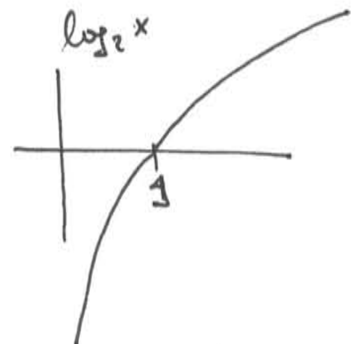
(a)
$$\int_0^n k^r \, dk \le \sum_{k=1}^n k^r \le \int_1^{n+1} k^r \, dk$$



$$\Rightarrow \quad \left.\frac{x^{r+1}}{r+1}\right|_0^n \le \sum_{k=1}^n k^r \le \left.\frac{x^{r+1}}{r+1}\right|_1^{n+1}$$

$$\Rightarrow \quad \frac{n^{r+1}}{r+1} \le \sum_{k=1}^n k^r \le \frac{(n+1)^{r+1}-1}{r+1}$$

(b)
$$\sum_{k=1}^n \log_2^s k$$



$$\therefore \quad \int_0^n \log_2^s x \, dx \le \sum_{k=1}^n \log_2^s k \le \int_1^{n+1} \log_2^s x \, dx$$

increasing for $\forall x \ge 1$

$$\therefore \text{ Bound } \quad \sum_{k=2}^n \log_2^s k \ge \int_1^n \log_2^s x \, dx$$

One would then integrate the expressions above by MMA

$$\int \ln^s x \, dx = -\Gamma(1+s, -\ln x)(-\ln x)^{-1-s}(\ln x)^{1+s}$$

07-06-022

(c)

$$\int_0^n x^r \log_2^s x\,dx \le \sum_{k=1}^n k^r \log_2^s k \le \int_1^{n+1} x^r \log_2^s x\,dx$$

But now this integral converges at $x=0$, due to the x^r factor.

By MMA: $\int x^r \ln^s x\,dx$

$$-\Gamma(1+s, -(1+r)\ln(x))\, \ln(x)^{1+s} \left(-(1+r)\ln(x)\right)^{-1-s}$$

pg 56 CLR

$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

$T(n) \overset{?}{\le} c\lfloor n/2 \rfloor + c\lceil \frac{n}{2} \rceil + 1 = cn + 1$

$T(n) \overset{?}{\le} (c\lfloor n/2 \rfloor - b) + (c\lceil n/2 \rceil - b) + 1$

$cn - 2b + 1$

$cn - b - b + 1 \le cn - b - (b-1) \le cn - b$ req $b \ge 1$

pg 56-87 CLR

~~For~~ eq 4.4

$T(n) = 2T(\lfloor n/2 \rfloor) + n$ Assm $T(n) \le cn$

$T(n) \le 2c\lfloor n/2 \rfloor + n \le cn + n \le (c+1)n \not\le cn$

- - -

$T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log_2 n$

let $m = \log_2 n$ $n = 2^m$

$\sqrt{n} = 2^{m/2}$

$T(2^m) = 2T(2^{m/2}) + m$

let $S(m) = T(2^m)$ $T(2^{m/2}) = S(m/2)$

$S(m) = 2S(m/2) + m$ $\Rightarrow$ $S(m) = O(m \log_2 m)$

04-06-021

$S(\log_2 n) = O(\log_2 n \log_2 \log_2 n)$

$\|$

$T(n) = O(\log_2 n \log_2(\log_2 n))$

(4.1-1) $T(n) = T(\lceil n/2 \rceil) + 1$

Assum $T(n) = O(\log_2 n) \rightarrow T(n) \leq c\log_2 n \quad \forall n \geq n_0$ some $c > 0$

~~Prove~~ $T(n) \leq c\log_2(n/2) + 1$

$= c\log_2(n) - c\log_2 2 + 1$

$= c\log_2 n - c + 1 \leq c\log_2 n$ iff $c \geq 1$

$\therefore T(n) = O(\log_2 n)$ How handel $\lceil\ \rceil$ specifically?

(4.1-2) Show: $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $\Omega(n\log_2 n)$

Assum $T(n) > c\,n\log_2 n \quad \forall n \geq n_0 \quad c > 0$

Then

$T(n) = 2T(\lfloor n/2 \rfloor) + n > 2c\lfloor n/2 \rfloor \log_2 \lfloor n/2 \rfloor + n$

ignoring floors $= 2c\frac{n}{2}\lg(n/2) + n = cn\lg(n/2) + n$

$= cn\lg n - cn\lg 2 + n$

$= cn\log_2 n - (c - 1)n \geq cn\log_2 n$ if $c > +1$.

- - - - Trying to do manipulations w/ floors in place.

$T(n) = 2T(\lfloor n/2 \rfloor) + n \geq 2c\lfloor n/2 \rfloor \log_2(\lfloor n/2 \rfloor) + n$

$\geq 2c(\frac{n}{2} - 1)\log_2(\frac{n}{2} - 1) + n$

$= cn\log_2(\frac{n}{2} - 1) - 2c\log_2(\frac{n}{2} - 1) + n$

$= cn\log_2(\frac{n-2}{2}) -$??

- - - - - - - - -

$\therefore \quad T(n) = \Omega(n\log_2 n)$.

(4.1-3) $T(1) = 1$ eq 4.4. $T(n) = 2T(\lfloor n/2 \rfloor) + n$

Claim $T(n) = O(n\log_2 n) \Rightarrow T(n) \leq Cn\log_2 n$

$T(1) \leq C \, 1 \log_2 1 = C \cdot 0$ No c can be chosen.

Assume $T(n) \leq cn\log_2 n + bn \quad \forall \; n \geq N$ New induction hypothesis

Then

$\cancel{T(n) \leq 2k\lfloor n/2 \rfloor} \; \cancel{2(c\lfloor n/2 \rfloor}$

$T(n) = 2T(\lfloor n/2 \rfloor) + n \leq 2\left(c\lfloor n/2 \rfloor \log_2 \lfloor n/2 \rfloor + bn\right) + n$

$\leq 2c\frac{n}{2}\log_2(n/2) + n(2b+1)$

$$= \cancel{2}\ cn\log_2 n - cn + n(2b+1)$$

$$= cn\log_2 n + (2b+1-c)n$$

$$\le cn\log_2 n \quad \text{iff} \quad c > 2b+1 \ \checkmark.$$

$+\ T(1) = 1 \overset{?}{\le} c\cdot 1\cdot 0 + b$

$\therefore$ pick $b = 1$ + $c \geq 5$ done

(4.1-4) $T(n) = \Theta(n\log_2 n) \Rightarrow c_1 n\log_2 n \le T(n) \le c_2 n\log_2 n$

$c_1, c_2 > 0 \ \forall n \ge n_0$

Pr by induction check

$c_1\cdot 0 \le T(1) = 1 \le c_2\cdot 0$ ... Ignoring B.C.'s

Assume $T(n)$ satisfies the above $\forall n \le N$.

Th $\cancel{T(2N)} \Leftarrow \cancel{T(}$

$T(n) = T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor) + \Theta(n)$

$\le c_2\lceil \frac{n}{2}\rceil \lg\lceil\frac{n}{2}\rceil +$

$c_2\lfloor\frac{n}{2}\rfloor \lg\lfloor n/2\rfloor + \Theta(n)$

$c_1\lceil\frac{n}{2}\rceil \lg\lceil\frac{n}{2}\rceil$ $\le$

$+ c_1\lfloor\frac{n}{2}\rfloor \lg\lfloor\frac{n}{2}\rfloor$

$+ \Theta(n)$

$\leq T(n) \leq c_2 \lceil \frac{n}{2} \rceil \lg \lceil \frac{n}{2} \rceil + c_2 \lceil \frac{n}{2} \rceil \lg \lceil \frac{n}{2} \rceil$

$+ \Theta(n)$

$2c_1 \lfloor \frac{n}{2} \rfloor \lg \lfloor \frac{n}{2} \rfloor + \Theta(n)$

$\leq 2c_2 \lceil \frac{n}{2} \rceil \lg \lceil \frac{n}{2} \rceil + \Theta(n)$

How do w/ floors & ceilings?

$\leq T(n) \leq$

$c_1 n \lg n - c_1 n + \Theta(n)$

$= c_2 n \lg n - c_2 n + \Theta(n)$

$\leq c_2 n \lg n$ if c_2 large enough

$c_1 n \lg n$ $\leq$

for c_1 large enough

4.1-5

$T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$ is $O(n \lg n)$

Assume $T(n) \leq c n \lg n$ $\forall n \geq N$.

pr. for larger n

$T(n) = 2T(\lfloor n/2 \rfloor + 17) + n \leq 2c(\lfloor \frac{n}{2} \rfloor + 17) \lg(\lfloor \frac{n}{2} \rfloor + 17) + n$

$$\therefore \quad T(n) \le 2c\left(\tfrac{n}{2}+17\right)\lg\left(\tfrac{n}{2}+17\right)+n$$

$$= cn\lg\left(\tfrac{n}{2}+17\right) + 34c\lg\left(\tfrac{n}{2}+17\right)+n$$

$$= cn\lg\left(\tfrac{n+34}{2}\right) + 34c\lg\left(\tfrac{n+34}{2}\right)+n$$

$$= cn\lg(n+34) - cn + 34c\lg(n+34) - 34c + n$$

$$= cn\lg(n+34) - (c-1)n + 34c\lg(n+34) - 34c$$

$$= cn\lg\left(n\left(1+\tfrac{34}{n}\right)\right) - (c-1)n + 34c\lg(n+34) - 34c$$

$$= cn\lg n + cn\lg\left(1+\tfrac{34}{n}\right) - (c-1)n + 34c\lg(n+34) - 34c$$

$$= cn\lg n + n\left[c\lg\left(1+\tfrac{34}{n}\right) - (c-1)\right] + 34c\lg(n+34) - 34c$$

$$\le cn\lg n + n\left[c\lg\left(1+\tfrac{34}{n}\right) - c + 1\right] + 34c\lg(n+34)$$

$$= cn\lg n + \left\{34c\lg(n+34) - n\left(c-1-c\lg\left(1+\tfrac{34}{n}\right)\right)\right\}$$

Now for large n & $c > 0$

$34c\lg(n+34) < n\left(c-1-c\lg\left(1+\tfrac{34}{n}\right)\right)$ Thus

$T(n) \le n\lg_2 n$.

4.1-6 $T(n) = 2T(\sqrt{n}) + 1$

let $n = 2^m$ $m = \lg n$

$T(2^m) = 2T(2^{m/2}) + 1$

let $S(m) = T(2^m)$

$S(m) = 2S(m/2) + 1 \Rightarrow S(m) = ?$

$\Rightarrow S(m) = \cancel{\Theta}\ O(m)$

$T(2^m) = O(m)$

$T(n) = O(\lg n)$

Check: Assume $T(n) \le c\lg n \cancel{-b}$

$T(n) = 2T(\sqrt{n}) + 1 \le 2(c\lg\sqrt{n} \cancel{-b}) + 1$

$= \frac{2c}{2}\lg n \cancel{+2b} + 1 = \cancel{c\lg n} + \cancel{(2b+1)}$

$\overset{?}{\le} \cancel{c\lg n - b} = c\lg n + 1 \overset{\text{No}}{\le} c\lg n +$

Same example as on pg 56 CLR.

pg 58 CLR

$$T(n) = n + 3\left\lfloor \frac{n}{4} \right\rfloor + 9\left\lfloor \frac{n}{16} \right\rfloor + 27\,T\left(\left\lfloor \frac{n}{64} \right\rfloor\right)$$

$$= n + 3\left\lfloor \frac{n}{4} \right\rfloor + 9\left\lfloor \frac{n}{16} \right\rfloor + \cdots + \qquad 3^{p}\,T(1)$$

$$\le n \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{k} + 3^{\log_4 n}\,\Theta(1) \qquad \text{w/ } p = \log_4 n$$

Now $3^{\log_4 n} = n^{\log_4 3}$

$$= n\left(\frac{1}{1-(3/4)}\right) + \Theta\left(n^{\log_4 3}\right)$$

$\cancel{\log_4 3 \ldots}$

$\log_4 3 < 1$

$$= n\,\frac{1}{\frac{1}{4}} + \Theta\left(n^{\log_4 3}\right)$$

$$= 4n + o(n) = O(n)$$

n^2

$\frac{1}{4}n^2 + \frac{1}{4}n^2 = \frac{1}{2}n^2$

$4\left(\frac{1}{4}\right)^2 n^2 = \frac{1}{4}n^2$

At $\left(\frac{n}{4^k}\right)$ when $k=$

$\left(\frac{n}{2^k}\right)$ at level k.

(4.2-1) $T(n) = 3T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + n$

$T(n) = 3\left(3T\left(\left\lfloor\frac{n}{4}\right\rfloor\right) + \left\lfloor\frac{n}{2}\right\rfloor\right) + n$

$= n + 3\left\lfloor\frac{n}{2}\right\rfloor + 9T\left(\left\lfloor\frac{n}{4}\right\rfloor\right) = n + 3\left\lfloor\frac{n}{2}\right\rfloor + 9\left(3T\left(\left\lfloor\frac{n}{8}\right\rfloor\right) + \left\lfloor\frac{n}{4}\right\rfloor\right)$

$= n + 3\left\lfloor\frac{n}{2}\right\rfloor + 9\left\lfloor\frac{n}{4}\right\rfloor + 27T\left(\left\lfloor\frac{n}{8}\right\rfloor\right)$ $3^3T\left(\left\lfloor\frac{n}{2^3}\right\rfloor\right)$

$= n + 3\left\lfloor\frac{n}{2}\right\rfloor + 9\left\lfloor\frac{n}{4}\right\rfloor + 27\left(3T\left(\left\lfloor\frac{n}{16}\right\rfloor\right) + \left\lfloor\frac{n}{8}\right\rfloor\right)$

$\begin{array}{r} 27 \\ 3 \\ \hline 60 \\ 21 \end{array}$

$= n + 3\left\lfloor\frac{n}{2}\right\rfloor + 9\left\lfloor\frac{n}{4}\right\rfloor + 27\left\lfloor\frac{n}{8}\right\rfloor + 81T\left(\left\lfloor\frac{n}{16}\right\rfloor\right)$ $3^4T\left(\left\lfloor\frac{n}{2^4}\right\rfloor\right)$

$\leq n + \left(\frac{3}{2}\right)n + \cancel{\frac{9}{4}}n \left(\frac{9}{4}\right)n + \frac{27}{8}n + 81T\left(\left\lfloor\frac{n}{16}\right\rfloor\right)$

$$\neq \quad n + \left(\tfrac{3}{2}\right)n + \left(\tfrac{9}{4}\right)n + \left(\tfrac{27}{8}\right)n + \cdots \quad 3^{p}T(1)$$

$$= n\sum_{i=0}^{p}\left(\tfrac{3}{2}\right)^{i} + 3^{p+1}T\left(\left\lfloor \tfrac{n}{2^{p+1}} \right\rfloor\right)$$

$p = 1, 2, 3, \ldots$

ends when

$2^{p+1} = n$

$p+1 = \lg n$

$p = \lg n - 1$

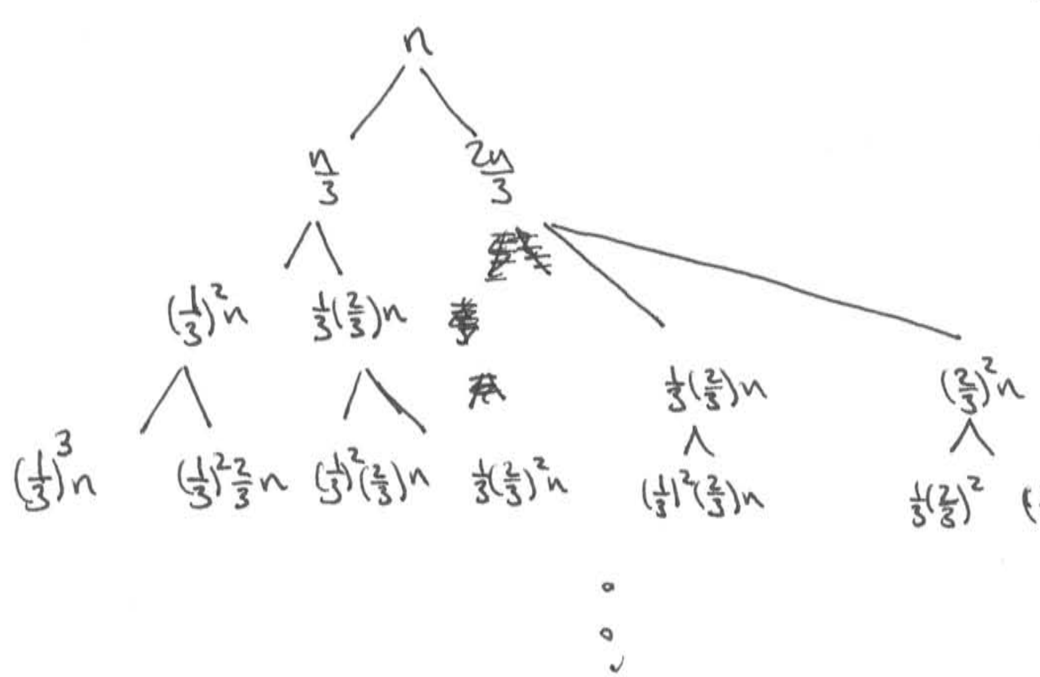
$$= 3^{\lg n}T(1) + n\sum_{i=0}^{\lg n - 1}\left(3/2\right)^{i}$$

$$= T(1)3^{\lg n} + n\left(\frac{1-(3/2)^{\lg n}}{1-(3/2)}\right)$$

$$= \cancel{T(1)3^{\lg n} +}$$

$$\sum_{k=0}^{n} x^{k} = \frac{1-x^{n+1}}{1-x}$$

4.2-2 $T(n) = T(n/3) + T(2n/3) + n$



Work	level
n	0
n	1
n	2
	$\vdots$

$$\left(\tfrac{1}{3}\right)^2 + \frac{2^2}{3^2} + \left(\tfrac{2}{3}\right)^2 = \frac{1}{9} + \frac{8}{9} = 1$$

\# of levels

$\Rightarrow \left(\tfrac{2}{3}\right)^{p} n = 1$

$p = \log_{3/2} n$

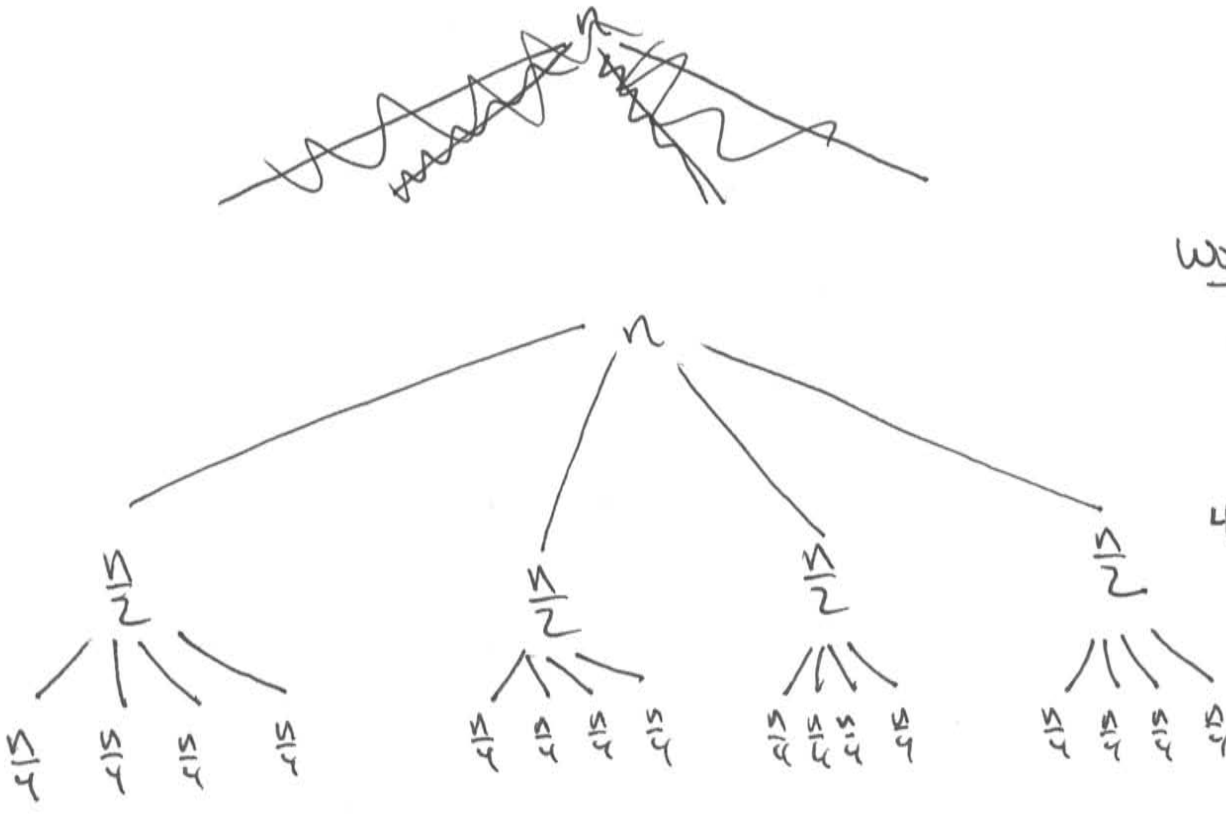
$\therefore \quad T(n) = \Theta(n \log_{3/2} n) = \Theta(n \lg n)$

$\log_{3/2} n = \dfrac{\lg n}{\lg 3/2}$ $\qquad \lg 3/2 = \lg(1.5) < 1$

$> \lg n$

$T(n) > n \lg n \Rightarrow T(n) = \Omega(n \lg n)$

4.2-3 $\quad T(n) = 4T(\lfloor n/2 \rfloor) + n$



Work	Level
n	0
$4(\frac{n}{2})$	1
$4^2(\frac{n}{4})$	2
$\vdots$	$\vdots$
$2^k n = 4^k(\frac{n}{2^k})$	k

Thus # of levels = $\frac{n}{2^k} = 1$

$$n = 2^k \Rightarrow k = \lg n$$

Thus $T(n) = \Theta\left(\sum_{k=0}^{\lg n} n 2^k\right) = \Theta\left(n \sum_{k=0}^{\lg n} 2^k\right)$

$$= \Theta\left(n\left(\frac{1-2^{\lg n+1}}{1-2}\right)\right) = \Theta\left(n\left(2^{\lg n+1}-1\right)\right)$$

$$= \Theta\left(n 2^{\lg n}\right)$$

4.2-4 $\quad T(n) = T(n-a) + T(a) + n$

$T(n) = \cancel{T(n-a) + T(a) + n} \quad n + T(a) + T(n-a)$ level #1

$= \cancel{T(n-2a)} + \cancel{2T(a) + n - a + n}$

level #2

$= n + T(a) + n - a + T(a) + T(n-2a) = 2n - a + 2T(a) + T(n-2a)$

$= n + T(a) + n - a + T(a) + (n-2a) + T(a) + T(n-3a)$

$\cancel{3n - 3a} +$

$= 3n - 3a + 3T(a) + T(n-3a)$ level #3

$= 3n - 3a + 3T(a) + n - 3a + T(a) + T(n-4a)$

$= 4n - 6a + 4T(a) + T(n-4a)$ level #4.

$\Rightarrow T(n) = l \cdot n - S_{l-1} a + l \cdot T(a) + T(n - la)$ w/ S_l defined on the next pg.

$1, 2, 3, 4, 5, \cdots$

$1, 3, 6, 10, \cdot \;=\; S_n \equiv \sum_{k=1}^{n} k \,. = \frac{n(n+1)}{2}$

$\overset{\parallel}{S_1} \; \overset{\parallel}{S_2} \; \overset{\parallel}{S_3} \; \overset{\parallel}{S_4}$

$\cdots$

go until $\quad n - la = 0 \Rightarrow l = \frac{n}{a}$

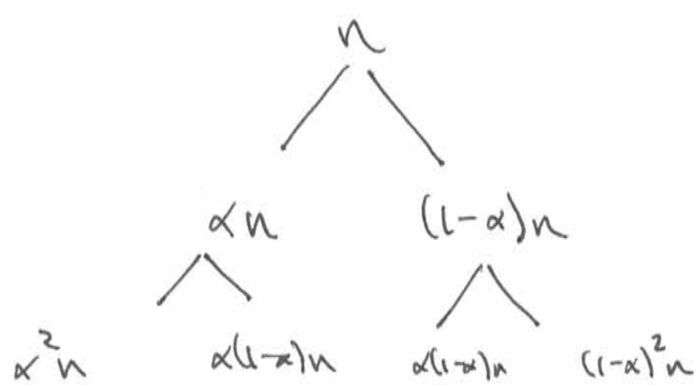
$$\therefore \; T(n) = \left(\frac{n}{a}\right)\cdot n - S_{\frac{n}{a}-1}\cdot a + \frac{n}{a}T(a) + T(0)$$

$$= \frac{n^2}{a} - \frac{\left(\frac{n}{a}-1\right)\left(\frac{n}{a}\right)}{2} + \frac{n}{a}T(a) + T(0)$$

$$\left\{ \sum_{k=0}^{n/a} (n - ka) = \right\}$$

$$T(n) = \frac{n^2}{a} - \frac{1}{2}\left(\frac{n^2}{a^2} - \frac{n}{a}\right) + \frac{n}{a}T(a) + T(0) = O(n^2)$$

(4.2-5) W.O.L.O.G assume ~~$\alpha \le 1/2$~~ $\alpha > 1/2$



level	work
0	n
1	n
2	n

$(1-\alpha+\alpha)^2 = (1-\alpha)^2 + 2\alpha(1-\alpha) + \alpha^2 =$

$\vdots$

\# of levels = Θ $\alpha^p n = 1$

$$n = \alpha^{-p}$$

$$-p = \log_\alpha n \Rightarrow p = -\log_\alpha n$$

Thus

$T(n) = \Theta(n \log n)$.

04-08-02

pg 64 CLR

4.3-1

(a) $T(n) = 4T(n/2) + n$

$a=4 \quad b=2 \quad \log_2 4 = 2$

Since $f(n) = n = O(\cancel{\log_2} n^{\log_2 4 - \epsilon}) \Rightarrow T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$

(b) $T(n) = 4T(n/2) + n^2$

$a=4 \quad b=2 \quad \log_2 4 = 2$

Now $f(n) = n^2 = \Theta(n^{\lg 4}) \Rightarrow T(n) = \cancel{\Theta \lg n} \; \Theta(\lg n \, n^2)$

(c) $T(n) = 4T(n/2) + n^3$

$a=4 \quad b=2 \quad \log_2 4 = 2$

Now $f(n) = n^3$

Since $f(n) = n^3 = \Omega(n^{2+\epsilon})$

Check $af(n/b) \leq cf(n)$

$4f(n/2) = 4\left(\frac{n}{2}\right)^3 = \frac{4}{8}n^3 = \frac{n^3}{2} \leq \cancel{\frac{3}{4}n^3} \; \frac{3}{4}n^3$ w/ $c = 3/2$

Then $T(n) = \Theta(n^3)$

4.3-2 $T(n) = 7T(n/2) + n^2 \qquad T'(n) = aT'(n/4) + n^2$

$a=7 \quad b=2$

$\log_2 7 = \frac{\ln 7}{\ln 2} = 2.807$

Now as $f(n) = \cancel{\Theta} \; O(n^{\log_2 7 - \epsilon}) \quad T(n) = \Theta(n^{\log_2 7})$

If $T'(n)$ is to be faster than $T(n) = \Theta(n^{\log_2 7})$

If ~~*~~ $\log_4 a = 2$ Then $T'(n) = \Theta(\log n \, n^2) = O(n^{\log_2 7})$

If $\log_4 a < 2$ Then $T'(n) = \Theta(n^2)$ assuming regularity condition.

If $\log_4 a > 2$ Then $T'(n) = \Theta(n^{\log_4 a})$

Thus as I increase a $T'(n)$ will increase the point at which it is the same complexity as $T(n)$ will be when

$$\log_4 a = \log_2 7$$

$$a = 4^{\log_2 7} \cong 49.0 .$$

4.3-3 $T(n) = T(n/2) + \Theta(1)$

$a = 1 \quad b = 2$

$\log_2 1 = 0$

Since $f(n) = \Theta(1) = \Theta(n^{\log_2 1}) = \Theta(1) \Rightarrow T(n) = \Theta(\lg n)$

4.3-4 We require $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\epsilon > 0$

But $a f(n/b) < c f(n)$ is Not true

pick a large + b small. $\log_b a$ is the large

$f(n) = n^{2.5}$ $\quad a = b^{2.5 - \delta}$ $\quad \log_b a = 2.5 - \delta$ ✓

Then $a\left(\frac{n}{b}\right)^{2.5} \overset{?}{<} c n^{2.5}$

$b^{2.5-\delta} \dfrac{n^{2.5}}{b^{2.5}} < c n^{2.5} \Rightarrow b^{-\delta} n^{2.5} < c n^{2.5} \Rightarrow b^{-\delta} < c < 1 ?$

Let $b > 1$

04-09-02 1

pg 65 CLR

$T(n) = f(n) + aT(n/b)$

$= f(n) + a\left(f(n/b) + aT(n/b^2)\right) = f(n) + af(n/b) + a^2T(n/b^2)$

$= f(n) + af(n/b) + a^2 f(n/b^2) + \cdots + a^k f(n/b^k)$ kth

$+ a^{k+1} T(n/b^{k+1})$ $1 \le k \le k$

when $\frac{n}{b^{k+1}} = 1$ This sum stops $\Rightarrow$ $n = b^{k+1}$

$k+1 = \log_b n$

$k = \log_b n - 1$

$= f(n) + af(n/b) + \cdots + a^{\log_b n - 1} f(n/b^{\log_b n - 1})$

$+ a^{\log_b n} T(1)$

Sim $a^{\log_b n} = n^{\log_b a}$

$$= \sum_{j=1}^{\log_b n - 1} a^j f(n/b^j) + \Theta(n^{\log_b a})$$

pg 67 CLR

1) $f(n) = O(n^{\log_b a - \epsilon})$ $f(n/b^j) = O((n/b^j)^{\log_b a - \epsilon})$

$$\therefore \quad g(n) = O\left(\sum_{j=0}^{\log_b n - 1} a^j \left(\frac{n}{b^j}\right)^{\log_b a - \epsilon}\right)$$

Now:

$$\sum_{j=0}^{\log_b n-1} a^j \left(\frac{n}{b^j}\right)^{\log_b a-\epsilon} = n^{\log_b a-\epsilon}\sum_{j=0}^{\log_b n-1}\left(\frac{a}{b^{\log_b a-\epsilon}}\right)^j$$

$$= n^{\log_b a-\epsilon}\sum_{j=0}^{\log_b n-1}\left(\frac{ab^\epsilon}{b^{\log_b a}}\right)^j \qquad b^{\log_b a} = a^{\log_b b} = a$$

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$= n^{\log_b a-\epsilon}\sum_{j=0}^{\log_b n-1}(b^\epsilon)^j = n^{\log_b a-\epsilon}\left(\frac{(b^\epsilon)^{\log_b n}-1}{b^\epsilon-1}\right)$$

$$= n^{\log_b a-\epsilon}\left(\frac{n^\epsilon-1}{b^\epsilon-1}\right) \qquad b^{\epsilon\log_b n} = b^{\log_b n^\epsilon} = n^\epsilon$$

$$= O(n^{\log_b a})$$

$$\Rightarrow \quad g(n) = O(n^{\log_b a}) \quad \checkmark$$

Case 2: $f(n) = \Theta(n^{\log_b a})$

$$f(n/b^j) = \Theta\left(\left(\frac{n}{b^j}\right)^{\log_b a}\right)$$

$$\therefore \quad g(n) = \Theta\left(\sum_{j=0}^{\log_b n-1} a^j \left(\frac{n}{b^j}\right)^{\log_b a}\right)$$

Now $$\sum_{j=0}^{\log_b n-1} a^j\left(\frac{n}{b^j}\right)^{\log_b a} = n^{\log_b a}\sum_{j=0}^{\log_b n-1}\left(\frac{a}{b^{\log_b a}}\right)^j$$

$$= n^{\log_b a} \sum_{j=0}^{\log_b n - 1} 1 = n^{\log_b a} \log_b n$$

Case 3: $a f(n/b) \le c f(n)$

$$a^j f(n/b^j) \le c^j f(n)$$

4.7 $$g(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) \qquad a \geq 1 \quad b > 1$$

$$g(n) = f(n) + \underbrace{\sum_{j=1}^{\log_b n - 1} a^j f(n/b^j)}_{\text{positive terms}}$$

$$\Rightarrow g(n) \geq f(n) \quad \forall n \qquad g(n) = \Omega(f(n))$$

Case 1:

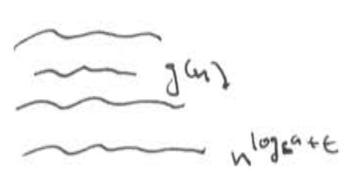
$$T(n) = \Theta(n^{\log_b a}) + O(n^{\log_b a}) = \Theta(n^{\log_b a})$$

Case 2:

$$T(n) = \Theta(n^{\log_b a}) + \Theta(n^{\log_b a} \lg n) = \Theta(n^{\log_b a} \lg n)$$

Case 3: $f(n) = \Omega(\cancel{\ } n^{\log_b a + \epsilon})$

$+$ since $g(n) = \Theta(f(n)) \Rightarrow g(n) = \Omega(n^{\log_b a + \epsilon})$



Then $T(n) = \Theta(n^{\log_b a}) + \Theta(g(n))$

$= \Theta(n^{\log_b a}) + \Theta(f(n))$

If $f(n) = \Omega(n^{\log_b a + \epsilon})$

$\Rightarrow T(n) = \Theta(f(n))$

$T_1(n) = aT_1(\lceil \frac{n}{b} \rceil) + f(n)$ $\qquad T_1 > T_2$

$T_2(n) = aT_2(\lfloor \frac{n}{b} \rfloor) + f(n)$

$\lceil x \rceil \leq x + 1$

$n_i = \begin{cases} n & i = 0 \\ \lceil \frac{n_{i-1}}{b} \rceil & i > 0 \end{cases}$

$n_0 = n$

$n_1 = \lceil \frac{n}{b} \rceil \leq \frac{n}{b} + 1$

$n_2 = \lceil \frac{n}{b^2} + \frac{1}{b} \rceil \leq \frac{n}{b^2} + \frac{1}{b} + 1$

$n_3 = \lceil \frac{n}{b^3} + \frac{1}{b^2} + \frac{1}{b} \rceil \leq \frac{n}{b^3} + \frac{1}{b^2} + \frac{1}{b} + 1$

pg 70-71 CLR

$$n_i = \begin{cases} n & \text{if } i = 0 \\ \left\lceil \frac{n_{i-1}}{b} \right\rceil & \text{if } i > 0 \end{cases}$$

So $n_0 = n$

$$n_1 = \left\lceil \frac{n_0}{b} \right\rceil = \left\lceil \frac{n}{b} \right\rceil \le \frac{n}{b} + 1$$

$$n_2 = \left\lceil \frac{n_1}{b} \right\rceil \le \frac{n_1}{b} + 1 \le \frac{1}{b}\left(\frac{n}{b}+1\right) + 1 = \frac{n}{b^2} + \frac{1}{b} + 1$$

$$n_3 = \left\lceil \frac{n_2}{b} \right\rceil \le \frac{n_2}{b} + 1 \le \frac{n}{b^3} + \frac{1}{b^2} + \frac{1}{b} + 1 \qquad \sum r^k = \frac{1}{1-r}$$

$\vdots$

$$n_j \le \frac{n}{b^j} + \sum_{k=0}^{j-1} \left(\frac{1}{b}\right)^k \le \frac{n}{b^j} + \frac{-1}{\frac{1}{b} - 1} = \frac{n}{b^j} + \frac{-b}{1-b}$$

$$= \frac{n}{b^j} + \frac{b}{b-1}$$

If $j = \lfloor \log_b n \rfloor$

$n = b^j \rightarrow j = \log_b n$

Then $\frac{n}{b^j} \le \frac{n}{b^{\log_b n}} = 1 \le b$

$$\therefore \quad n_j \le b + \frac{b}{b-1} = \frac{b^2 - b + b}{b-1} = \frac{b^2}{b-1} = O(1)$$

$$T(n) = aT(\lceil \tfrac{n}{b} \rceil) + f(n)$$

$$T(n_0) = f(n_0) + aT(\lceil \tfrac{n_0}{b} \rceil) = f(n_0) + aT(n_1)$$

$$= f(n_0) + a[f(n_1) + aT(n_2)] = f(n_0) + af(n_1) + a^2 f(n_2) + a^3 T(n_3)$$

$$= f(n_0) + af(n_1) + \cdots a^{\lfloor \log_b n \rfloor - 1} f(n_{\lfloor \log_b n - 1 \rfloor})$$

$$+ a^{\lfloor \log_b n \rfloor} T(n_{\lfloor \log_b n \rfloor})$$

Now ~~$\lfloor \log_b n \rfloor \leq \log_b n + 1$~~

$\therefore$ ~~$a^{\lfloor \log_b n \rfloor} \leq a a^{\log_b n} = a n^{\log_b a}$~~

$$\log_b n - 1 \leq \lfloor \log_b n \rfloor \leq \log_b n + 1$$

$$a^{-1} n^{\log_b a} \leq a^{\lfloor \log_b n \rfloor} \leq a n^{\log_b a}$$

Thus $\quad a^{\lfloor \log_b n \rfloor} = \Theta(n^{\log_b a})$

Pg 71 CLR 04-11-02 1

$$a f(\lceil n/b \rceil) \leq c f(n)$$

$$\Rightarrow \quad a^j f(n_j) \leq c^j f(n)$$

$$\therefore \sum_{j=0}^{\lfloor \log_b n \rfloor - 1} a^j f(n_j) = O(f(n))$$

Case 2: $f(n) = \Theta(n^{\log_b a})$ show $f(n_j) = O\left(\frac{n^{\log_b a}}{a^j}\right) = O\left(\left(\frac{n}{b^j}\right)^{\log_b a}\right)$

$b^{\log_b a} = a^1$ $\qquad j \leq \lfloor \log_b n \rfloor \qquad b^j/n < 1$

$\therefore a^j = b^{j \log_b a} =$

$f(n) = \Theta(n^{\log_b a})$

$f(n_j) \leq c n_j^{\log_b a}$ for n ~~large~~ enough

$$\leq c\left(\frac{n}{b^j} + \frac{b}{b-1}\right)^{\log_b a} = c\left(\frac{n}{b^j}\right)^{\log_b a}\left(1 + \frac{b^j}{n}\left(\frac{b}{b-1}\right)\right)^{\log_b a}$$

$$= c\left(\frac{n^{\log_b a}}{b^{j\log_b a}}\right)\left(1 + \frac{b^j}{n}\left(\frac{b}{b-1}\right)\right)^{\log_b a}$$ if $j \leq \lfloor \log_b n \rfloor$, $\frac{b^j}{n} < 1$

$$\leq c\,\frac{n^{\log_b a}}{a^j}\left(1 + \frac{b}{b+1}\right)^{\log_b a}$$

$$\leq O\left(\frac{n^{\log_b a}}{a^j}\right)$$

Pg 72 CLR

4.4-1 eq 4.12

$$n_i = \begin{cases} n & \text{if } i=0 \\ \left\lfloor \frac{n_{i-1}}{b} \right\rfloor & \text{if } i>0 \end{cases}$$

From pages 32 + 33 for integers n + integers a + b

$$\left\lfloor \left\lfloor \frac{n}{a} \right\rfloor \frac{1}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor$$

Thus $n_0 = n$ $\quad n_1 = \left\lfloor \frac{n}{b} \right\rfloor$ $\quad n_2 = \left\lfloor \frac{1}{b} \left\lfloor \frac{n}{b} \right\rfloor \right\rfloor = \left\lfloor \frac{n}{b^2} \right\rfloor$

... $\quad n_i = \left\lfloor \frac{n}{b^i} \right\rfloor$

4.4-2 If $f(n) = \Theta(n^{\log_b a} \lg^k n)$

We have shown that

$$T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$$

This sum looks like

$$\sum_{j=0}^{\log_b n-1} a^j f(n/b^j) = \Theta\left(\sum_{j=0}^{\log_b n-1} \cancel{a^j} \frac{n^{\log_b a}}{\cancel{(b^j)^{\log_b a}}} \lg^k\left(\frac{n}{b^j}\right) \right)$$

$$= \Theta\left(\sum_{j=0}^{\log_b n-1} n^{\log_b a} \lg^k\left(\frac{n}{b^j}\right) \right)$$

$$= \cancel{\Theta(n^{\log_b a}} \;\Theta\left(n^{\log_b a} \sum_{j=0}^{\log_b n-1} \lg^k n - \lg^k b^j \right)$$

$$\lg(n/b^j) = \lg n - \lg b^j$$

$$(\lg(n/b^j))^k = (\qquad)^k$$

$$= \Theta\left(n^{\log_b a}\left(\lg^k(n) + \lg^k(n/b) + \lg^k(n/b^2) + \cdots + \lg^k(n/b^{\log_b n-1})\right)\right)$$

$$= \cancel{\Theta\left(n^{\log_b a} \lg^{k+1} n\right)}$$

This becomes a problem w/ how to evaluate the following sum.

$$\sum_{j=0}^{\log_b n-1} \lg^k(n/b^j)$$

Assume $n = b^p$, then this sum becomes: An exact power of b as suggested in the text.

$$\sum_{j=0}^{p-1} \lg^k(b^{p-j})$$

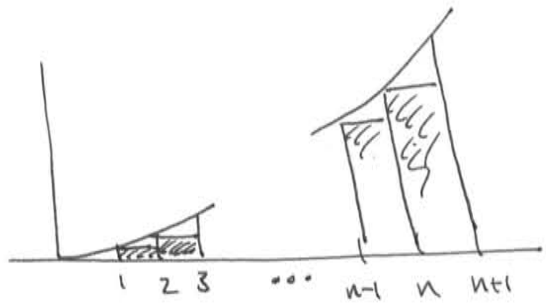
$$\lg b^{p-j} = (p-j)\lg b$$

$$\lg^k b^{p-j} = (p-j)^k \lg^k b$$

$$= \sum_{j=0}^{p-1} (p-j)^k \lg^k b$$

$$= \lg^k b \sum_{j=0}^{p-1} (p-j)^k = \lg^k b\left(p^k + (p-1)^k + (p-2)^k + \cdots + 1\right)$$

$$= \lg^k b \sum_{i=1}^{p} i^k \quad . \quad \text{since } i^k \text{ is an increasing fn of } i$$



$$\therefore \sum_{i=1}^{p} i^k \le \int_1^{p+1} x^k \, dx$$

$$= \left. \frac{x^{k+1}}{k+1} \right|_1^{p+1} = \frac{\left((p+1)^{k+1} - 1\right)}{k+1}$$

Thus $\lg^k b \sum_{i=1}^{p} i^k \le \frac{\lg^k b}{k+1} \left((p+1)^{k+1} - 1 \right)$

since $p = \log_b n$ $\quad = \frac{\lg^k b}{k+1} \left((\log_b n + 1)^{k+1} - 1 \right)$

$$= \frac{\lg^k b}{k+1} \left[\log_b^{k+1} n \left(1 + \frac{1}{\log_b n}\right)^{k+1} - 1 \right]$$

$1 + \frac{1}{\log_b n} \le 2$

$$\le \frac{\lg^k b}{k+1} \left[2^{k+1} \log_b^{k+1} n - 1 \right]$$

Now $\log_b n = \frac{\lg n}{\lg b}$ $\quad \therefore$ Above $= O(\lg^{k+1} n)$ ✓.

Pg 72 CLR

04-11-02

1

4.4-3 $\quad af(n/b) \le cf(n) \Rightarrow \exists\ \epsilon > 0 \ni$

$f(n) = \Omega(n^{\log_b a + \epsilon})$

$a(n/b)^p \le cn^p$

$\frac{a}{b^p} \cancel{n^p} = c\cancel{n^p}$

$\Rightarrow a = cb^p$

$\approx$ if $c = \Theta(1)$

$p = \log_b a$.

Given a fixed n, then

$\cancel{af(n/b) \le cf(n)}$

$f(n) \geq (\frac{a}{c})f(n/b)$

Thus $\quad f(n) \geq (\frac{a}{c})f(n/b) \geq (\frac{a}{c})^2 f(n/b^2) \geq (\frac{a}{c})^3 f(n/b^3) \geq (\frac{a}{c})^4 f(n/b^4)$

$\ldots \quad (\frac{a}{c})^i f(n/b^i)$

Thus $\quad a^i f(n/b^i) \le c^i f(n)$

$f(n) \geq (\frac{a}{c})^i f(n/b^i) \qquad$ let $i = \lceil \log_b n \rceil$

$\geq (\frac{a}{c})^{\lceil \log_b n \rceil} f(n/b^{\lceil \log_b n \rceil})$

Now $\quad \lceil \log_b n \rceil \le \log_b n + 1 \quad \therefore$ the cbac

$\geq \frac{a^{\lceil \log_b n \rceil}}{c^{\lceil \log_b n \rceil}} f(n/n \cdot b) \geq \frac{a^{\log_b n - 1}}{c^{\log_b n + 1}} f(b)$

Thus $f(n) \geq \dfrac{a^{-1} n^{\log_b a}}{c\, n^{\log_b c}} f(b) = \dfrac{f(b)}{a \cdot c} \dfrac{n^{\log_b a}}{n^{\log_b c}}$

Now $c < 1$ & $b > 1$ so

$\log_b c < 0$ let $-\epsilon = \log_b c$

Thus $f(n) \geq \dfrac{f(b)}{ac} \dfrac{n^{\log_b a}}{n^{-\epsilon}} = \dfrac{f(b)}{ac} n^{\log_b a + \epsilon}$

$\rightarrow f(n) = \Omega(n^{\log_b a + \epsilon})$

(4-1)

(a) $T(n) = 2T(n/2) + n^3$

$$T(n) = n^3 + 2T(n/2)$$

$$= n^3 + 2\left(\left(\tfrac{n}{2}\right)^3 + 2T(n/4)\right) = n^3 + 2\left(\tfrac{n}{2}\right)^3 + 4T(n/4)$$

$$= n^3 + 2\left(\tfrac{n}{2}\right)^3 + 4\left(\left(\tfrac{n}{4}\right)^3 + 2T(n/8)\right)$$

$$= n^3 + 2\left(\tfrac{n}{2}\right)^3 + 4\left(\tfrac{n}{4}\right)^3 + 8T(n/8)$$

$$= n^3 + 2\left(\tfrac{n}{2}\right)^3 + 4\left(\tfrac{n}{4}\right)^3 + 8\left(\left(\tfrac{n}{8}\right)^3 + 2T(n/16)\right)$$

$$= n^3 + 2\left(\tfrac{n}{2}\right)^3 + 4\left(\tfrac{n}{4}\right)^3 + 8\left(\tfrac{n}{8}\right)^3 + 16T(n/16)$$

$$= n^3 + 2\left(\tfrac{n}{2}\right)^3 + 4\left(\tfrac{n}{4}\right)^3 + 8\left(\tfrac{n}{8}\right)^3 + \cdots + 2^{\lfloor \lg n \rfloor - 1}\left(\frac{n}{2^{\lfloor \lg n \rfloor - 1}}\right)^3 + 2^{\lfloor \lg n \rfloor} T\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right)$$

stops when $\frac{n}{2^p} = 1$

$n = 2^p$

$p = \lg n$

$=$

$$= n^3 + \frac{n^3}{2^2} + \frac{n^3}{4^2} + \frac{n^3}{8^2} + \cdots + \frac{n^3}{(2^{\lfloor \lg n \rfloor - 1})^2} + 2^{\lfloor \lg n \rfloor} T\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right)$$

$$= n^3 + \frac{n^3}{2^2} + \frac{n^3}{2^4} + \frac{n^3}{2^6} + \cdots + \frac{n^3}{2^{2\lfloor \lg n \rfloor - 2}} + 2^{\lfloor \lg n \rfloor} T\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right)$$

$$= n^3\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \cdots + \frac{1}{2^{2\lfloor \lg n \rfloor - 2}}\right) + 2^{\lfloor \lg n \rfloor}\, T\!\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right)$$

$$= n^3\,\Theta(1) + 2^{\lfloor \lg n \rfloor}\, T\!\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right)$$

↑ Since this sum converges

Now: $\cancel{\lg n \leq \lfloor \lg n \rfloor \leq \lg n + 1}$

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil \leq x + 1$$

so

$$\lg n - 1 < \lfloor \lg n \rfloor < \lg n$$

$$\therefore\ 2^{-1} \cdot 2^{\lg n} < 2^{\lfloor \lg n \rfloor} < 2^{\lg n} \Rightarrow \frac{1}{2^{\lg n}} < \frac{1}{2^{\lfloor \lg n \rfloor}} < \frac{2}{2^{\lg n}}$$

$$+\ 2^{\lg n} = 2^{\log_2 n} = n \Rightarrow \frac{1}{n} < \frac{1}{2^{\lfloor \lg n \rfloor}} < \frac{2}{n}$$

$$+\ \cancel{\frac{n}{\lfloor \lg n \rfloor}} \qquad \cancel{\frac{n}{\lfloor \lg n \rfloor} < \frac{n}{\lg}}$$

Thus

$$1 < \frac{n}{2^{\lfloor \lg n \rfloor}} < 2 \quad \therefore\ T\!\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right) = \Theta(1)$$

$$+\ 2^{\lfloor \lg n \rfloor} = \Theta(n)$$

$$\therefore\ T(n) = \Theta(n^3).$$

(b) $T(n) = T\left(\frac{9n}{10}\right) + n$

$$T(n) = n + T\left(\frac{9n}{10}\right)$$

$$= n + 1\left[\frac{9n}{10} + T\left(\left(\frac{9}{10}\right)^2 n\right)\right] = n + \frac{9n}{10} + T\left(\left(\frac{9}{10}\right)^2 n\right)$$

$$= n + \frac{9}{10}n + \left(\frac{9}{10}\right)^2 n + T\left(\left(\frac{9}{10}\right)^3 n\right)$$

$$= 1n + \left(\frac{9}{10}\right)n + \left(\frac{9}{10}\right)^2 n + \cdots$$

will be 1 when $\left(\frac{9}{10}\right)^p n = 1$

$$n = \left(\frac{10}{9}\right)^p \equiv r^p \quad \text{w/} \quad r \equiv \frac{10}{9}$$

$$p = \log_r n$$

$$= n + \left(\frac{9}{10}\right)n + \left(\frac{9}{10}\right)^2 n + \cdots + n\left(\frac{9}{10}\right)^{\lfloor \log_r n \rfloor - 1} + T\left(\left(\frac{9}{10}\right)^{\lfloor \log_r n \rfloor} \cdot n\right)$$

$$= n \sum_{k=0}^{\lfloor \log_r n \rfloor - 1} \left(\frac{9}{10}\right)^k + T\left(\left(\frac{9}{10}\right)^{\lfloor \log_r n \rfloor} \cdot n\right)$$

Now $= n\,\Theta(1)$

since $\sum$ converges

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

$$\log_r n - 1 < \lfloor \log_r n \rfloor \leq \log_r n$$

SO

$$\left(\tfrac{9}{10}\right)^{\log_r n} \le \left(\tfrac{9}{10}\right)^{\lfloor \log_r n \rfloor} < \left(\tfrac{9}{10}\right)^{\log_r n - 1}$$

$$n^{\log_r\left(\frac{9}{10}\right)} \le \left(\tfrac{9}{10}\right)^{\lfloor \log_r n \rfloor} < \tfrac{10}{9} \cdot n^{\log_r\left(\frac{9}{10}\right)}$$

$$\log_r\left(\tfrac{9}{10}\right) = \log_{\frac{10}{9}} \tfrac{9}{10} = -1$$

$$\therefore \quad n^{-1} \le \left(\tfrac{9}{10}\right)^{\lfloor \log_r n \rfloor} \le \tfrac{10}{9}\, n^{-1}$$

Multiply by n $\Rightarrow$

$$1 \le n\left(\tfrac{9}{10}\right)^{\lfloor \log_r n \rfloor} \le \tfrac{10}{9}$$

$$\therefore \quad T\left(n\left(\tfrac{9}{10}\right)^{\lfloor \log_r n \rfloor}\right) = \Theta(1)$$

$$\therefore \quad T(n) = \Theta(n)$$

(c) $T(n) = n^2 + 16T(n/4)$

$$= n^2 + 16\left[\left(\frac{n}{4}\right)^2 + 16T\left(\frac{n}{4^2}\right)\right] = n^2 + 4^2\left(\frac{n}{4}\right)^2 + 4^4 T\left(\frac{n}{4^2}\right)$$

$$= n^2 + n^2 + 4^4\left[\frac{n^2}{4^4} + 4^2 T\left(\frac{n}{4^3}\right)\right]$$

$$= n^2 + n^2 + n^2 + 4^6 T\left(\frac{n}{4^3}\right)$$

$$= 3n^3 + 4^6\left[\frac{n^2}{4^6} + 4^2 T\left(\frac{n}{4^4}\right)\right] = \cancel{3n}\ 4n^2 + 4^8 T\left(\frac{n}{4^4}\right)$$

$$= kn^2 + 4^{2k} T\left(\frac{n}{4^k}\right) \qquad k = 1, 2, 3, \ldots, ?$$

this expansion ends when $\frac{n}{4^k} = 1 \Rightarrow n = 4^k$

$\log_4 n = k$

$$\Leftarrow \lfloor \log_4 n \rfloor n^2 + 4^{2\lfloor \log_4 n \rfloor} T\left(\frac{n}{4^{\lfloor \log_4 n \rfloor}}\right)$$

Now

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

So $\log_4 n - 1 \leq \lfloor \log_4 n \rfloor \leq \log_4 n$

$\therefore$ $(\log_4 n - 1) n^2 \leq \lfloor \log_4 n \rfloor n^2 \leq (\log_4 n) n^2$

$$\cancel{\phi}\;\; 4^{2((\log_4 n)-1)} \leq 4^{2\lfloor \log_4 n \rfloor} \leq 4^{2(\log_4 n)}$$

$$\frac{1}{8}\, 4^{2\log_4 n} \leq 4^{2\lfloor \log_4 n \rfloor} \leq \left(4^{\log_4 n}\right)^2$$

$$\frac{1}{8} n^2 \leq \quad \leq n^2$$

$$\frac{1}{4^{\log_4 n}} \leq \frac{1}{4^{\lfloor \log_4 n \rfloor}} \leq \frac{1}{4^{(\log_4 n - 1)}}$$

$$\frac{1}{n} \leq \frac{1}{4^{\lfloor \log_4 n \rfloor}} \leq \frac{4}{n}$$

$$\therefore \quad T(n/4^{\lfloor \log_4 n \rfloor}) = \Theta(1).$$

$$\therefore \quad T(n) = \Theta((\log_4 n)\, n^2) = \Theta(n^2 \lg n)$$

Check w/ master thm: $f(n) = n^2$ $\quad a = 16 \quad b = 4$

$\log_b a = \log_4 16 = 2$

(d) $T(n) = 7T(n/3) + n^2$

$$= n^2 + 7T(n/3)$$

$$= n^2 + 7\left[\left(\frac{n}{3}\right)^2 + 7T\left(\frac{n}{9}\right)\right] = n^2 + 7\left(\frac{n}{3}\right)^2 + 7^2 T\left(\frac{n}{3^2}\right)$$

$$= n^2 + 7\left(\frac{1}{3}\right)^2 n^2 + 7^2\left[\frac{n^2}{3^4} + 7T\left(\frac{n}{3^3}\right)\right]$$

$$= n^2 + 7\left(\frac{1}{3}\right)^2 n^2 + 7^2\left(\frac{1}{3}\right)^4 n^2 + 7^3 T\left(\frac{n}{3^3}\right) \qquad p = \log_3 n$$

$$= n^2 + 7\left(\frac{1}{3}\right)^2 n^2 + 7^2\left(\frac{1}{3}\right)^4 n^2 + \cdots + 7^{\lfloor \log_3 n \rfloor - 1}\left(\frac{1}{3}\right)^{2(\lfloor \log_3 n \rfloor - 1)} n^2$$

$$+ 7^{\lfloor \log_3 n \rfloor} T\left(\frac{n}{3^{\lfloor \log_3 n \rfloor}}\right)$$

$$= n^2 \sum_{k=0}^{\lfloor \log_3 n \rfloor - 1} \left(\frac{7}{9}\right)^k + 7^{\lfloor \log_3 n \rfloor} T\left(\frac{n}{3^{\lfloor \log_3 n \rfloor}}\right)$$

$$= \Theta(n^2)$$

$$\begin{array}{ccccc} 7^{-1} 7^{\log_3 n} & \le & 7^{\lfloor \log_3 n \rfloor} & \le & 7^{\log_3 n} \\ \| & & & & \| \\ 7^{-1} n^{\log_3 7} & \le & '' & \le & n^{\log_3 7} \end{array}$$

& From below $T\left(\frac{n}{3^{\lfloor \log_3 n \rfloor}}\right) = \Theta(1)$

Now $\log_3 7 = \frac{\ln 7}{\ln 3} = 1.77$

sin $1.77 < 2$

$T(n) = \Theta(n^2)$.

(e) $T(n) = 7T(n/2) + n^2$

$= n^2 + 7T(n/2)$

$= n^2 + 7\left[\left(\frac{n}{2}\right)^2 + 7T(n/4) \right] = n^2 + 7\left(\frac{n}{2}\right)^2 + 7^2 T\left(\frac{n}{2^2}\right)$

$= n^2 + 7\left(\frac{1}{2}\right)^2 n^2 + 7^2 \left(\left(\frac{n}{2^2}\right)^2 + 7T\left(\frac{n}{2^3}\right)\right)$

$= n^2 + 7\left(\frac{1}{2}\right)^2 n^2 + 7^2 \left(\frac{1}{2^4}\right) n^2 + 7^3 T(n/2^3)$

$= n^2 + 7\left(\frac{1}{2}\right)^2 n^2 + 7^2\left(\frac{1}{2^4}\right) n^2 + 7^3 \left[\frac{n^2}{2^6} + 7T(n/2^4) \right]$

$= n^2 + \left(\frac{\sqrt{7}}{2}\right)^2 n^2 + \left(\frac{\sqrt{7}}{2}\right)^4 n^2 + \left(\frac{\sqrt{7}}{2}\right)^6 n^2 + 7^4 T(n/2^4)$

This can be continued until $\frac{n}{2^p} = 1 \Rightarrow n = 2^p$
$p = \lg n$

$\therefore$

$$T(n) = n^2 \sum_{k=0}^{\lfloor \lg n \rfloor - 1} \left(\frac{\sqrt{7}}{2}\right)^k + 7^{\lfloor \lg n \rfloor} T(n/2^{\lfloor \lg n \rfloor})$$

(4-1) (e) Now $x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$

~~$\lfloor \lg n \rfloor$~~

So $\lg n - 1 < \lfloor \lg n \rfloor \le \lg n$

Thus

$$2^{\lg n - 1} < 2^{\lfloor \lg n \rfloor} \le 2^{\lg n}$$

$$\frac{n}{2} < 2^{\lfloor \lg n \rfloor} \le n$$

$$\frac{1}{n} \le \frac{1}{2^{\lfloor \lg n \rfloor}} < \frac{2}{n}$$

$\therefore \quad 1 \le \dfrac{n}{2^{\lfloor \lg n \rfloor}} < 2$ so $T(n/2^{\lfloor \lg n \rfloor}) = \Theta(1)$

&

$$7^{\lg n - 1} \le 7^{\lfloor \lg n \rfloor} \le 7^{\lg n} = 7^{\log_2 n} = n^{\log_2 7}$$

$$7^{\lg n - 1} = 7^{-1} n^{\log_2 7}$$

$$\log_2 7 = \frac{\ln 7}{\ln 2} = 2.807$$

Now $\displaystyle\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$ so that $\displaystyle\sum_{k=0}^{\lfloor \lg n \rfloor - 1} r^k = \frac{1-r^{\lfloor \lg n \rfloor}}{1-r}$

$$\therefore \quad T(n) = n^2\left(\frac{1-(\sqrt{7}/2)^{\lfloor \lg n \rfloor}}{1-(\sqrt{7}/2)}\right) + \Theta(n^{\log_2 7})$$

$$= n^2\left(\frac{(\sqrt{7}/2)^{\lfloor \lg n \rfloor} - 1}{(\sqrt{7}/2) - 1}\right) + \Theta(n^{\lg 7})$$

$$= \frac{n^2 (\sqrt{7}/2)^{\lfloor \lg n \rfloor}}{\sqrt{7}/2 - 1} - \frac{n^2}{\sqrt{7}/2 - 1} + \Theta(n^{\lg 7})$$

$$\left(\frac{\sqrt{7}}{2}\right)^{\lg n - 1} < \left(\frac{\sqrt{7}}{2}\right)^{\lfloor \lg n \rfloor} < \left(\frac{\sqrt{7}}{2}\right)^{\lg n}$$

$$\frac{2}{\sqrt{7}} n^{\log_2(\sqrt{7}/2)} < \left(\frac{\sqrt{7}}{2}\right)^{\lfloor \lg n \rfloor} < n^{\log_2(\sqrt{7}/2)}$$

Now $\log_2(\sqrt{7}/2) = \dfrac{\ln(\sqrt{7}/2)}{\ln 2} = .4036$

$$\therefore \quad \left(\frac{\sqrt{7}}{2}\right)^{\lfloor \lg n \rfloor} = \Theta(n^{\lg(\sqrt{7}/2)}) = \Theta(n^{.4036})$$

sin $\lg 7 = 2.8$

& $2 + .4036 < \lg 7$

$T(n) = \Theta(n^{\lg 7})$

Prob 4-1 $\quad$ Pg 72 CLR $\quad$ 04-15-02

(f) $T(n) = \sqrt{n} + 2T(n/4)$ $\quad k=1$

$= \sqrt{n} + 2\left[\frac{\sqrt{n}}{2} + 2T\left(\frac{n}{4^2}\right)\right] = \sqrt{n} + \sqrt{n} + 2^2 T\left(\frac{n}{4^2}\right)$ $\quad k=2$

$= \cancel{2\sqrt{n} + 2^2 T(\tfrac{n}{4^2})} = 2\sqrt{n} + 2^2\left[\frac{\sqrt{n}}{4} + 2T(n/4^3)\right]$

$= 3\sqrt{n} + 2^3 T(n/4^3)$ $\quad k=3$

$\vdots$

$= k\sqrt{n} + 2^k T(n/4^k) \quad \forall \quad 1 \le k \le \lfloor \log_4 n \rfloor$

$= \lfloor \log_4 n \rfloor \sqrt{n} + 2^{\lfloor \log_4 n \rfloor} T\left(\frac{n}{4^{\lfloor \log_4 n \rfloor}}\right)$

Now $\quad \log_4 n - 1 \le \lfloor \log_4 n \rfloor \le \log_4 n$

Thus

$$2^{-1} 2^{\log_4 n} \le 2^{\lfloor \log_4 n \rfloor} \le 2^{\log_4 n}$$
$$\parallel$$
$$2^{-1} n^{\log_4 2} \le 2^{\lfloor \log_4 n \rfloor} \le n^{\log_4 2}$$
$$\parallel$$
$$2^{-1} n^{1/2} \le 2^{\lfloor \log_4 n \rfloor} \le n^{1/2}$$

& From before $\quad \frac{n}{4^{\lfloor \log_4 n \rfloor}} = \Theta(1) \quad \therefore T(n) = \Theta(\log_4 n \, n^{1/2}) + \Theta(n^{1/2})$

$= \Theta(\lg n \, n^{1/2})$

(g) $T(n) = T(n-1) + n$

$\Rightarrow T(n+1) - T(n) = n+1$

$\Delta_n T = n+1$

$\sum \Rightarrow T(n) - T(1) = \sum_{k=1}^{n+1} (k+1) = \Theta(n^2)$

(h) $T(n) = T(\sqrt{n}) + 1$

$T(n) = 1 + T(\sqrt{n}) = 1 + 1 + T(n^{1/4}) = 2 + T(n^{1/4})$ $(n^{1/2})^{1/2} = n^{1/4}$ $k=2$

$= 3 + T(n^{1/8})$ ... $k=3$

$\vdots$

$= k + T(n^{1/2^k})$ $1 \le k \le p$ w/ p?

$n^{(1/2^p)} = 1$

In general $n^{\frac{1}{2^i}} = 1$ after an ∞ # of square roots. As some point $n^{1/2^p} \approx 1$ or close to 1 & we can take

$T(n^{\frac{1}{2^p}}) = \Theta(1)$ or constant work. Thus to simplify the derivation

Assume $n = 2^{2^N}$ Then $n^{\frac{1}{2^p}} = (2^{2^N})^{\frac{1}{2^p}} = 2^{\frac{2^N}{2^p}}$ → let

let $p = N$.

$\therefore \quad T(n) = \Theta(N) = \cancel{\Theta(}$

$\Rightarrow \quad n = 2^{2^N} \Rightarrow \quad \lg n = 2^N \Rightarrow \quad N = \lg(\lg n)$ &

$T(n) = \lg(\lg n)$ how do we show this for arbitrary n?

(4-2)

$A[1, \ldots, n] \longrightarrow B[0, \ldots, n]$

How to implement in $O(n)$ time using

$B[A[1]] = 1$

$B[A[2]] = 1$

$\vdots$

$B[A[n]] = 1$

$\hookrightarrow$ After n opperations one of $B[k]$ will still be zero. Assuming that it was initalized to 0. looking through this B array will find the 0th missing element if $B[k] = 0$ return k.

~~I'm going to assume that the elts of A are sorted.~~

Don't fully understand the problem?

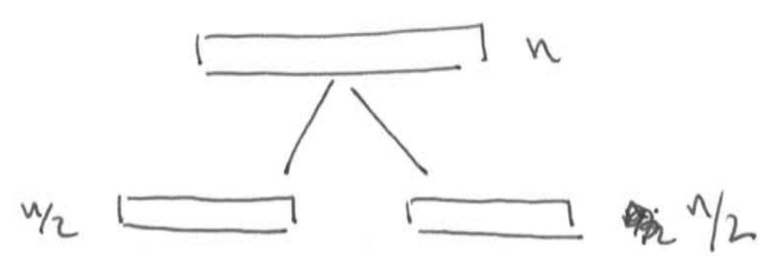
Is the array A sorted or unsorted?

(4-3) Note: I do (b) 1st + then (a).

(b) $T(n) = \begin{cases} 2 & n = 2 \\ 2T(n/2) + n & n = 2^k \quad k > 1 \end{cases}$ is functional equation for worst case running time.

w/ 3 different methods

I assume that on 1 recursion call we pass 2 arrays of length $n/2$



Thus the new recursion eq becomes

$$T(n) = 2T(n/2) \quad + \quad n \quad + 2 \text{ passing costs}$$

to solve two ~~sub~~ problems of size $n/2$

to merge two sorted arrays each of length $n/2$.

to pass sub arrays to the subroutine

I would think this would be a two way cost, one to pass to the subroutines & once when brought back from those subroutines.

∴ I'll only use a 1 for one way passing costs:

$$T(n) = 2T(n/2) + n + \begin{cases} \Theta(1) & (1) \\ \Theta(n) & (2) \\ \Theta(n) & (3) \end{cases}$$

⇒ ~~Thus~~ = Case (1), (2), (3) make no difference on ~~#~~ in the complexity calculation & thus the ~~runn~~ worst case running time does not change

⇒ $T(n) = \cancel{\Theta(\lg n)}$ $\Theta(n\lg n)$

~~For~~ ~~Merge-sort~~ ~~(1.3.1)~~

(a) For Binary search in a sorted array. Since this can be done in place there is no passing of arrays & thus suffers no penalties from such.

Pg 73 CLR

(4-4)

(a) $T(n) = 3T(n/2) + n\lg n$

$$T(n) = n\lg n + 3T(n/2) = n\lg n + 3\left[\frac{n}{2}\lg\left(\frac{n}{2}\right) + 3T\left(\frac{n}{4}\right)\right]$$

$$= n\lg n + 3\left(\frac{n}{2}\right)\lg\left(\frac{n}{2}\right) + 3^2 T\left(\frac{n}{2^2}\right)$$

$$= n\lg n + 3\left(\frac{n}{2}\right)\lg\left(\frac{n}{2}\right) + 3^2\left[\left(\frac{n}{2^2}\right)\lg\left(\frac{n}{2^2}\right) + 3T\left(\frac{n}{2^3}\right)\right]$$

$$= n\lg n + 3\left(\frac{n}{2}\right)\lg\left(\frac{n}{2}\right) + 3^2\left(\frac{n}{2^2}\right)\lg\left(\frac{n}{2^2}\right) + 3^3 T(n/2^3)$$

$$= n\lg n + 3\left(\frac{n}{2}\right)\lg\left(\frac{n}{2}\right) + 3^2\left(\frac{n}{2^2}\right)\lg\left(\frac{n}{2^2}\right) + \cdots 3^k\left(\frac{n}{2^k}\right)\lg\left(\frac{n}{2^k}\right)$$

$$+ 3^{k+1} T(n/2^{k+1}) \qquad \forall \quad 1 \le k \le \lg n - 1$$

$$\frac{n}{2^{k+1}} = 1$$

$$n = 2^{k+1}$$

$$k+1 = \lg n$$

$\therefore$ let $k = \lfloor \lg n \rfloor - 1$

$\therefore$

$$T(n) = n\lg n + 3\left(\frac{n}{2}\right)\lg\left(\frac{n}{2}\right) + 3^2\left(\frac{n}{2^2}\right)\lg\left(\frac{n}{2^2}\right) + \cdots + 3^{\lfloor \lg n \rfloor - 1}\left(\frac{n}{2^{\lfloor \lg n \rfloor - 1}}\right) \cdot \lg\left(\frac{n}{2^{\lfloor \lg n \rfloor - 1}}\right)$$

$$+ 3^{\lfloor \lg n \rfloor} T\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right)$$

Now $\lg\left(\frac{n}{2^p}\right) = \lg n - p\lg 2 = \lg n - p$

$\therefore$

$$T(n) = n\lg n + 3\left(\frac{n}{2}\right)\left[\lg n - 1\right] + 3^2\left(\frac{n}{2^2}\right)\left[\lg n - 2\right] + \cdots +$$

$$3^{\lfloor \lg n\rfloor - 1}\left(\frac{n}{2^{\lfloor \lg n\rfloor - 1}}\right)\left[\lg n - (\lfloor \lg n\rfloor - 1)\right] + 3^{\lfloor \lg n\rfloor} T\left(\frac{n}{2^{\lfloor \lg n\rfloor}}\right)$$

$$\Rightarrow T(n) = \sum_{k=0}^{\lfloor \lg n\rfloor - 1}\left(\frac{3}{2}\right)^k n\left[\lg n - k\right] + 3^{\lfloor \lg n\rfloor} T\left(\frac{n}{2^{\lfloor \lg n\rfloor}}\right)$$

$\lg n = \lfloor \lg n\rfloor + \epsilon$ $\quad$ $\epsilon > 0$ & $\epsilon < 1$

$$\Rightarrow T(n) = \sum_{k=0}^{\lfloor \lg n\rfloor - 1}\left(\frac{3}{2}\right)^k n\left[\lfloor \lg n\rfloor + \epsilon - k\right] + 3^{\lfloor \lg n\rfloor} T\left(\frac{n}{2^{\lfloor \lg n\rfloor}}\right)$$

$\Big\{$ let $k_2 = \lfloor \lg n\rfloor - k$ $\quad$ $k=0$ $\; k_2 = \lfloor \lg n\rfloor$ $\Big\}$

$\quad\quad k = \lfloor \lg n\rfloor - 1$ $\; k_2 = +1$

$$T(n) = \underbrace{\epsilon\sum_{k=0}^{\lfloor \lg n\rfloor - 1}\left(\frac{3}{2}\right)^k n} + \sum_{k=0}^{\lfloor \lg n\rfloor - 1}\left(\frac{3}{2}\right)^k n\left[\lfloor \lg n\rfloor - k\right] + 3^{\lfloor \lg n\rfloor} T\left(\frac{n}{2^{\lfloor \lg n\rfloor}}\right)$$

$$\epsilon n\left(\frac{1 - (3/2)^{\lfloor \lg n\rfloor}}{1 - 3/2}\right) + \sum_{k=0}^{\lfloor \lg n\rfloor - 1}\left(\frac{3}{2}\right)^k n\left[\lfloor \lg n\rfloor - k\right] + 3^{\lfloor \lg n\rfloor} T\left(\frac{n}{2^{\lfloor \lg n\rfloor}}\right)$$

For 2nd sum let $k_2 = \lfloor \lg n \rfloor - k$ $\hookrightarrow$ $k = \lfloor \lg n \rfloor - k_2$

Then when $k = 0$ $k_2 = \lfloor \lg n \rfloor$

$k = \lfloor \lg n \rfloor - 1$ $k_2 = +1$

$\therefore$

$$T(n) = 2cn\left(\left(\tfrac{3}{2}\right)^{\lfloor \lg n \rfloor} - 1\right) + \sum_{k_2=+1}^{\lfloor \lg n \rfloor} \left(\tfrac{3}{2}\right)^{\lfloor \lg n \rfloor - k_2} \cdot n\, k_2 + 3^{\lfloor \lg n \rfloor} T\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right)$$

$$= 2cn\left(\left(\tfrac{3}{2}\right)^{\lfloor \lg n \rfloor} - 1\right) + n\left(\tfrac{3}{2}\right)^{\lfloor \lg n \rfloor} \underbrace{\sum_{k=1}^{\lfloor \lg n \rfloor} \left(\tfrac{2}{3}\right)^{k} k}_{?} + 3^{\lfloor \lg n \rfloor} T\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right)$$

Since

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x} = \frac{x^{n+1} - 1}{x - 1}$$

$$\sum_{k=0}^{\lfloor \lg n \rfloor} \left(\tfrac{2}{3}\right)^k k$$

Add zero term.

$$\sum_{k=0}^{n} k x^{k-1} = \frac{(n+1)x^n(x-1) - (x^{n+1} - 1)}{(x-1)^2} = \frac{(n+1)(x^{n+1} - x^n) - x^{n+1} + 1}{(x-1)^2}$$

$$= \cdots = \frac{n x^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

Thus

$$\sum_{k=1}^{\lfloor \lg n \rfloor} \left(\tfrac{2}{3}\right)^k k = \frac{\lfloor \lg n \rfloor (2/3)^{\lfloor \lg n \rfloor + 1} - (\lfloor \lg n \rfloor + 1)(2/3)^{\lfloor \lg n \rfloor} + 1}{(2/3 - 1)^2} = \Theta\left(\lg n \left(\tfrac{2}{3}\right)^{\lg n}\right)$$

Now $\left(\frac{2}{3}\right)^{\lg n} = n^{\lg(2/3)} = n^{\frac{\ln 2/3}{\ln 2}} = n^{-.58}$

& $T\left(\frac{n}{2^{\lfloor \lg n \rfloor}}\right) = T(1) = \Theta(1)$

$\therefore$

$T(n) = \Theta\left(n\left(3/2\right)^{\lg n}\right) + \Theta\left(n\left(3/2\right)^{\lg n} \cdot \lg n \cdot \left(\frac{2}{3}\right)^{\lg n}\right)$

$+ \Theta\left(3^{\lg n}\right)$

$= \cancel{\Theta(n(3/2)^{\lg n})+} \; \Theta\left(n \cdot n^{\overset{\lg 3 - \lg 2}{\overset{\|}{\lg(3/2)}}}\right) + \Theta\left(n^{\lg 3}\right)$

$= \Theta\left(n^{1+\frac{\ln(3/2)}{\ln 2}}\right) + \Theta\left(n^{\frac{\ln 3}{\ln 2}}\right)$

$= \Theta\left(n^{1.58\ldots}\right) + \Theta(\quad = \quad \Theta\left(n^{1.58\ldots}\right).$

pg 73 CLR 04-29-02 1

4-4

(b) $T(n) = 3T\left(\frac{n}{3} + 5\right) + \frac{n}{2}$

$$= \frac{n}{2} + 3T\left(\frac{n}{3} + 5\right) \qquad k=0$$

$$= \frac{n}{2} + 3\left[\frac{1}{2}\left(\frac{n}{3} + 5\right) + 3T\left(\frac{n}{3^2} + \frac{5}{3} + 5\right)\right]$$

$$= \frac{n}{2} + 3\frac{1}{2}\left(\frac{n}{3} + 5\right) + 3^2T\left(\frac{n}{3^2} + \frac{5}{3} + 5\right) \qquad k=1$$

$$= \frac{n}{2} + \frac{3}{2}\left(\frac{n}{3} + 5\right) + 3^2\left[\frac{1}{2}\left(\frac{n}{3^2} + \frac{5}{3} + 5\right) + 3T\left(\frac{n}{3^3} + \frac{5}{3^2} + \frac{5}{3} + 5\right)\right]$$

$$= \cancel{\frac{n}{2} + \frac{3}{2}\left(\frac{n}{3} + 5\right) + 3^2\left[\frac{1}{2}\left(\frac{n}{3^2} + \frac{5}{3} + 5\right)\right.}$$

$$= \frac{n}{2} + \frac{3}{2}\left(\frac{n}{3} + 5\right) + \frac{3^2}{2}\left(\frac{n}{3^2} + \frac{5}{3} + 5\right) + 3^3T\left(\frac{n}{3^3} + \frac{5}{3^2} + \frac{5}{3} + 5\right) \qquad k=2$$

$$= \frac{n}{2} + \frac{3}{2}\left(\frac{n}{3} + 5\right) + \frac{3^2}{2}\left(\frac{n}{3^2} + \frac{5}{3} + 5\right) + \frac{3^3}{2}\left(\frac{n}{3^3} + \frac{5}{3^2} + \frac{5}{3} + 5\right)$$

$$+ 3^4T\left(\frac{n}{3^4} + \frac{5}{3^3} + \frac{5}{3^2} + \frac{5}{3} + 5\right) \qquad k=3$$

$$= \ldots$$

$$= \cancel{\frac{n}{2}} +$$

$$\frac{n}{2} + \frac{n}{2} + \frac{3}{2}\cdot 5 + \frac{n}{2} + \frac{3(5)}{2} + \frac{3^2}{2}5$$

$$+ \frac{n}{2} + \frac{3\cdot 5}{2} + \frac{3^2\cdot 5}{2} + \frac{3^3}{2}\cdot 5 +$$

$$= \frac{n}{2} + \frac{n}{2} + \frac{n}{2} + \frac{n}{2}$$

$$+ \cancel{\frac{5}{2}(3} + 3$$

$$= \frac{n}{2} + \frac{3}{2}\left(\frac{n}{3} + 5\right) + \frac{3^2}{2}\left(\frac{n}{3^2} + 5\left(1 + \frac{1}{3}\right)\right)$$

$$+ \frac{3^3}{2}\left(\frac{n}{3^3} + 5\left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\right)$$

$$+ \cdots \frac{3^k}{2}\left(\frac{n}{3^k} + 5\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{k-1}}\right)\right)$$

$$+ 3^{k+1} T\left(\frac{n}{3^{k+1}} + 5\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k}\right)\right)$$

$$0 \le k \le$$

$$\sum_{k=0}^{N} \left(\frac{1}{3}\right)^k = \frac{1-(1/3)^{N+1}}{1-(1/3)} = \frac{1-(1/3)^{N+1}}{(2/3)}$$

$$= \frac{3}{2}\left(1-(1/3)^{N+1}\right) = \Theta(3/2)$$

Thus $$\frac{n}{3^{k+1}} + 5\sum_{k=0}^{k}\left(\frac{1}{3}\right)^k = \frac{n}{3^{k+1}} + \Theta\left(\frac{15}{2}\right) = \Theta(1)$$

$$\Rightarrow \quad \frac{n}{3^{k+1}} = \Theta(1)$$

$$k+1 = \log_3 n \qquad k = \log_3 n - 1$$

Then

$$T(n) = \frac{n}{2} + \frac{3}{2}\left(\frac{n}{3}+5\right) + \frac{3^2}{2}\left(\frac{n}{3^2}+5\left(1+\frac{1}{3}\right)\right)$$

$$+ \frac{3^3}{2}\left(\frac{n}{3^3}+5\left(1+\frac{1}{3}+\frac{1}{3^2}\right)\right) + \cdots$$

$$+ \frac{3^k}{2}\left(\frac{n}{3^k}+5\sum_{p=0}^{k-1}\left(\frac{1}{3}\right)^p\right) + \underbrace{3^{\log_3 n}\Theta(1)}_{=\,n\,\Theta(1)}$$

$$T(n) = \frac{n}{2} + \frac{3^{-1}3^{\log_3 n}}{2} \cdot 5 \sum_{p=0}^{k-1} \left(\frac{1}{3}\right)^p$$

$$+ \Theta(n)$$

$$= \Theta(n)$$

(4-4)

(c) $T(n) = 2T(n/2) + \frac{n}{\lg n}$

$$T(n) = \frac{n}{\lg n} + 2T(n/2)$$

$$= \frac{n}{\lg n} + 2\left[\frac{n/2}{\lg(n/2)} + 2T(n/2^2)\right]$$

$$= \frac{n}{\lg n} + \frac{n}{\lg(n/2)} + 2^2 T(n/2^2)$$

$$= \frac{n}{\lg n} + \frac{n}{\lg(n/2)} + 2^2\left[\frac{n/2^2}{\lg(n/2^2)} + 2T(n/2^3)\right]$$

$$= \frac{n}{\lg n} + \frac{n}{\lg(n/2)} + \frac{n}{\lg(n/2^2)} + 2^3 T(n/2^3)$$

$$= \frac{n}{\lg n} + \frac{n}{\lg(n/2)} + \frac{n}{\lg(n/2^2)} + \cdots + \frac{n}{\lg(n/2^{k-1})}$$

$$+ 3^k T(n/2^k) \qquad 1 \le k \le K$$

01-29-02 2

w/ I + $\frac{n}{2^I} = \Theta(1)$

$$I = \lg n$$

$\therefore$

$$T(n) = \sum_{p=0}^{\lg n} \frac{n}{\lg(n/2^p)} + 3^{\lg n}\Theta(1)$$

$$n^{\lg 3}\Theta(1)$$

$$= n \sum_{p=0}^{\lg n} \frac{1}{\lg(n/2^p)}$$

$$\sum_{p=0}^{\lg n} \frac{1}{\lg n - p} = \sum_{p=0}^{\lg n} \frac{1}{p}$$

Dropping $p=0$ term $\sum_{p=1}^{\lg n} \frac{1}{p} = \Theta(\lg \lg n)$

$$\therefore T(n) = \Theta(n \lg\lg n) + \Theta(n^{\lg 3})$$

$$\lg 3 = \log_2 3 = \frac{\ln 3}{\ln 2} = 1.58$$

Since $n^{\lg 3} = \Omega(n \lg \lg n) \Rightarrow T(n) = \Theta(n^{\lg 3})$

(4-4)

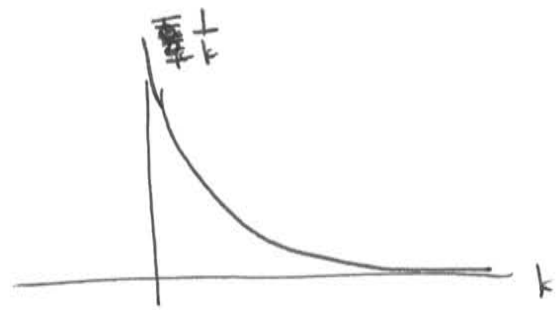
(d) $T(n) = T(n-1) + \frac{1}{n}$

$$= \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + T(n-3)$$

$2 = n-k$
$k = n-2$

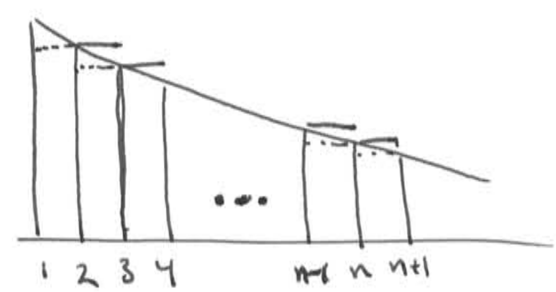
$$= \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{2} + T(n-(n-1))$$

$$= \sum_{k=2}^{n} \frac{1}{k} + T(1)$$



Now since $1/k$ is decreasing

$$\int_{x=2}^{n+1} \frac{dx}{x} \leq \sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{dx}{x}$$



$$\Rightarrow \ln x \Big|_{2}^{n+1} \leq \sum_{k=2}^{n} \frac{1}{k} \leq \ln n$$

$\ln(n+1) - \ln 2 \leq$

$$\therefore \sum_{k=2}^{n} \frac{1}{k} = \Theta(\ln n) = \Theta(\lg n)$$

$$\therefore T(n) = \Theta(\lg n)$$

4-4

(e) $T(n) = T(n-1) + \lg n$

$$T(n) - T(n-1) = \lg n$$

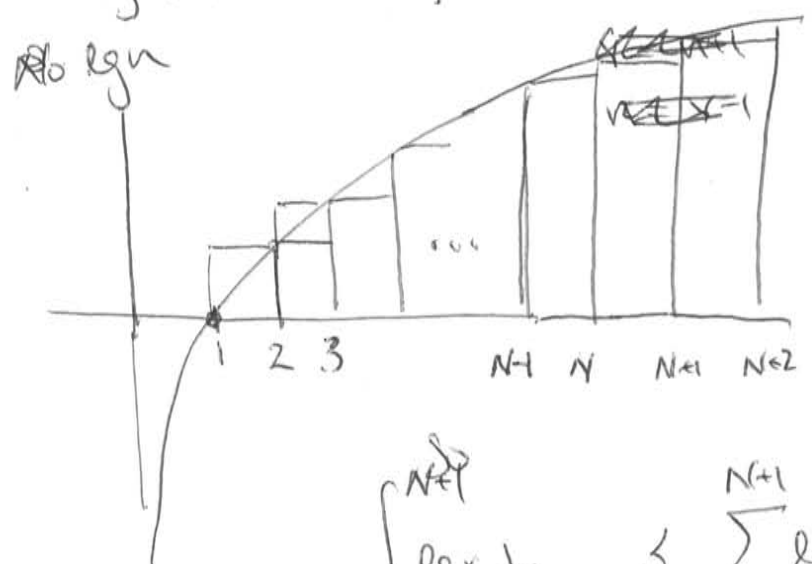
$$T(n+1) - T(n) = \lg(n+1)$$

$$\Delta T(n) = \lg(n+1) \qquad \sum_{n=1}^{N}$$

$$T(n)\Big|_{1}^{N+1} = \sum_{n=1}^{N} \lg(n+1)$$

$$\Rightarrow T(N+1) - T(1) = \sum_{n=1}^{N} \lg(n+1) = \sum_{n=2}^{N+1} \lg(n)$$

Now $\lg n$ is an increasing fn



$$\int_{1}^{N+1} \lg x \, dx \le \sum_{n=2}^{N+1} \lg n \le \int_{2}^{N+2} \lg x \, dx$$

cont

$$\lg n = \frac{\ln x}{\ln 2}$$

~~ln x - 1~~ + 1

$$+ \int_{x_1}^{x_2} \ln x \, dx = x \ln x - x \Big|_{x_1}^{x_2}$$

So $$\frac{1}{\ln 2}(x \ln x - x)\Big|_1^{N+1} \le \sum_{n=2}^{N+1} \lg n \le \frac{1}{\ln 2}(x \ln x - x)\Big|_2^{N+2}$$

$$= \frac{1}{\ln 2}\Big((N+1)\ln(N+1)\Big) \le \sum_{n=2}^{N+1} \lg n \le \frac{1}{\ln 2}\Big((N+2)\ln(N+2) - (N+2) - 2\ln 2 + 2\Big)$$

$$\therefore \sum_{n=2}^{N+1} \lg n = \Theta(N \lg N)$$

So $T(n) = \Theta(N \lg N)$

pg 74 CLR

04-30-02 1

(4-4)

(f) $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$

$$T(n) = n + \sqrt{n}\, T(\sqrt{n})$$

$$= n + n^{1/2} T(n^{1/2}) \qquad k = 1$$

$$= n + n^{1/2}\left[n^{1/2} + n^{1/4}\, T(n^{1/4}) \right]$$

$$= n + n + n^{\frac{1}{2}+\frac{1}{4}} T(n^{1/4}) \qquad k = 2$$

$$= 2n + n^{\frac{1}{2}+\frac{1}{4}} \left[n^{1/4} + n^{1/8}\, T(n^{1/8}) \right]$$

$$= \cancel{\quad}$$

$$= 2n + n + n^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} T(n^{1/8})$$

$$= 3n + n^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} T(n^{1/8}) \qquad k = 3$$

$$= kn + n^{\sum_{p=1}^{k} (\frac{1}{2})^p} T(n^{1/2^k})$$

$$\sum_{p=1}^{k} (\tfrac{1}{2})^p \le \frac{1}{(1-1/2)} - 1$$

$$= 1$$

~~TR~~: $1 \le k \le k$

This stops when $n^{\frac{1}{2^k}} \cong \Theta(1)$

②

say $n^{\frac{1}{2^k}} = 2$

$\frac{1}{2^k} \lg n = 1$

$\Rightarrow 2^k = \lg n \Rightarrow k = \cancel{\lg(\lg n)} \lg(\lg n)$

Thus $T(n) = \lg(\lg n) \cdot n + n^{\sum_{p=1}^{\lg(\lg n)} (1/2)^p} T(2)$

$= \omega(n^3)$

$\therefore T(n) = \Theta(n \cdot \lg(\lg n))$

4-5

(a) $T(n)$ + $h(n)$ monotonically increasing functions

$$T(n) \leq h(n) \quad \forall\, n = b^p \quad b > 1$$

$h(n)$ is slowly growing $h(n) = O(h(n/b)) \Rightarrow$

$\exists$ constants c + n_0 $\ni$

$$h(n) \leq c\,h(n/b) \quad \forall\, n \geq n_0.$$

$$\leq c^p h(n/b^p) \,.\, \forall \quad p = 1, 2, \cdots, \lfloor \log_b n \rfloor \quad \{ c > 1$$

$h(n/b^p) \approx h(1)$ for $p = \lfloor \log_b n \rfloor$

Now if ~~f(n)~~ $n = b^p$ $p = 1, 2 \ldots$

$$T(n) \leq h(n) \leq c^p h(n/b^p) = c^p \quad \forall\, n \ni$$

$$b^p \leq n \leq b^{p+1}. \quad ?$$

- - - - - -

(b) $T(n) = aT(n/b) + f(n)$

$$T(n_0) = aT(n_0/b) + f(n_0) = a g(n_0/b) + f(n_0)$$

(4-6) Pg 74 CLR 04-06-02 1

$$F(z) = \sum_{i=0}^{\infty} F_i z^i = F_0 + F_1 z + F_2 z^2 + \dots$$

& ~~ten~~ $F_i = F_{i-1} + F_{i-2} \quad i \geq 2 \quad F_0 = 0, \; F_1 = 1$

(a) $F(z) = F_0 + F_1 z + \sum_{i=2}^{+\infty} F_i z^i$

$$= \cancel{F_0}^{0} + \cancel{F_1}^{1} z + \sum_{i=2}^{+\infty} (F_{i-1} + F_{i-2}) z^i$$

$$= z + \sum_{i=2}^{+\infty} F_{i-1} z^i + \sum_{i=2}^{+\infty} F_{i-2} z^i$$

$$= z + \sum_{i=1}^{+\infty} F_i z^{i+1} + \sum_{i=0}^{\infty} F_i z^{i+2}$$

$$= z + z F(z) + z^2 F(z)$$

(b) $\rightarrow$

$$0 = +z + (z-1) F(z) + z^2 F(z) \rightarrow \quad F(z) = \frac{-z}{-1+z+z^2}$$

$$\rightarrow F(z) = \frac{z}{1-z-z^2}$$

(c) $\cancel{1-z-z^2} = \cancel{z}$

$$z^2+z-1 = (z-z_1)(z-z_2) = z^2-(z_1+z_2)z+z_1z_2$$

w/ $z_1 = \dfrac{-1+\sqrt{1+4}}{2}$ $\qquad$ $z_2 = \dfrac{-1-\sqrt{1+4}}{2}$

$= \dfrac{-1+\sqrt{5}}{2}$ $\qquad$ $= \dfrac{-1-\sqrt{5}}{2}$

Check $z_1+z_2 = -1$ yes

$z_1 \cdot z_2 = -1$

$$= \frac{+1+\cancel{\sqrt{5}}-\cancel{\sqrt{5}}-5}{4} = -1 \quad \checkmark$$

$\therefore$

$$F(z) = \frac{-z}{(z-z_1)(z-z_2)} = \frac{-z}{\left(\frac{z}{z_1}-1\right)\left(\frac{z}{z_2}-1\right)z_1(z_2)}$$

$$= \frac{z}{(1-z/z_1)(1-z/z_2)}$$

Now $\dfrac{1}{z_1} = \dfrac{2}{-1+\sqrt{5}} = \dfrac{2}{-1+\sqrt{5}} \cdot \dfrac{-1-\sqrt{5}}{-1-\sqrt{5}} = \dfrac{2(-1-\sqrt{5})}{1-5}$

$$= \frac{-(1+\sqrt{5})}{-2} = \frac{1+\sqrt{5}}{2} \equiv \phi$$

$$\& \ \frac{1}{z_2} = \frac{2}{-1-\sqrt{5}}\left(\frac{-1+\sqrt{5}}{-1+\sqrt{5}}\right) = \frac{2(-1+\sqrt{5})}{+1-5} = \frac{2(-1+\sqrt{5})}{-4}$$

$$= \frac{1-\sqrt{5}}{2} \equiv \hat{\phi}$$

$$\therefore \ F(z) = \frac{z}{(1-\phi z)(1-\hat{\phi} z)} = \frac{A}{1-\phi z} + \frac{B}{1-\hat{\phi} z}$$

$$\frac{z}{1-\hat{\phi} z} = A + \frac{B(1-\phi z)}{1-\hat{\phi} z}\Bigg|_{z=1/\phi} \Rightarrow \frac{1/\phi}{1-\hat{\phi}/\phi} = A$$

$$\Rightarrow A = \frac{1}{\phi-\hat{\phi}} = \frac{1}{\frac{1+\sqrt{5}}{2}-\left(\frac{1-\sqrt{5}}{2}\right)}$$

$$= \frac{1}{\sqrt{5}}$$

$$\& \ \frac{z}{1-\phi z} = \frac{A(1-\hat{\phi} z)}{1-\phi z} + B\Bigg|_{z=1/\hat{\phi}}$$

$$\frac{1/\hat{\phi}}{1-\phi/\hat{\phi}} = B \Rightarrow B = \frac{1}{\hat{\phi}-\phi} = \frac{-1}{\sqrt{5}}$$

$$\therefore \ F(z) = \frac{1}{\sqrt{5}}\left[\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right]$$

(c) Thus $F(z) = \frac{1}{\sqrt{5}}\left[\sum_{k=0}^{+\infty} \phi^k z^k - \sum_{k=0}^{+\infty} \hat{\phi}^k z^k\right]$

$$= \sum_{k=0}^{\infty} \frac{1}{\sqrt{5}}(\phi^k - \hat{\phi}^k) z^k$$

(d) $\therefore F_i = \frac{1}{\sqrt{5}}(\phi^k - \hat{\phi}^k)$

so since $|\hat{\phi}| < 1$ $\quad \hat{\phi}^k \xrightarrow[k \to +\infty]{} 0$

$|\hat{\phi}| < 1$

$F_k = \frac{\phi^k}{\sqrt{5}} - \frac{\hat{\phi}^k}{\sqrt{5}}$

Since the sign of $\hat{\phi}^k$ can be both positive or negative depending on whether k is a power of two or not

F_k is always slightly (less than 1) away from $\frac{\phi^k}{\sqrt{5}}$

(e) $F_{i+2} = \frac{1}{\sqrt{5}}(\phi^{i+2} - \hat{\phi}^{i+2}) = \phi^i \left(\frac{\phi^2 - (\hat{\phi}\phi)^i \hat{\phi}^2}{\sqrt{5}}\right)$

Since $\hat{\phi}\phi = -1$

$F_{i+2} = \phi^i\left(\frac{\phi^2 - (-1)^i \hat{\phi}^2}{\sqrt{5}}\right)$

01-06-02 5

i even $\qquad \dfrac{\phi^2 - \hat{\phi}^2}{\sqrt{5}} = \dfrac{\left(\dfrac{\cancel{1}+2\sqrt{5}+\cancel{5}}{4} - \dfrac{(\cancel{1}-2\sqrt{5}+\cancel{5})}{4}\right)}{\sqrt{5}}$

i odd $\qquad \dfrac{\phi^2 + \hat{\phi}^2}{\sqrt{5}} = \dfrac{\left(\dfrac{1+\cancel{2\sqrt{5}}+5}{4} + \dfrac{1-\cancel{2\sqrt{5}}+5}{4}\right)}{\sqrt{5}}$

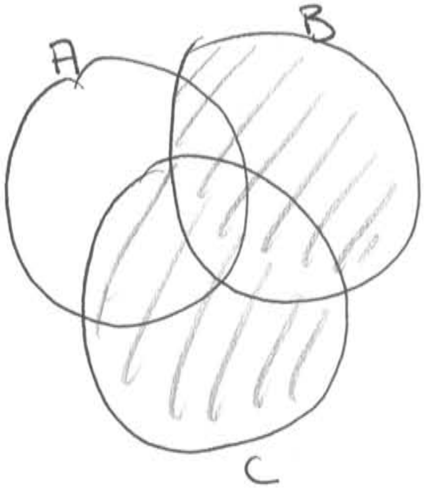
i even $= \dfrac{\dfrac{4\sqrt{5}}{4}}{\sqrt{5}} = 1$

i odd $\qquad \dfrac{\dfrac{2(6)}{4}}{\sqrt{5}} = \dfrac{3}{\sqrt{5}} > 1 \qquad 2 < \sqrt{5} < 3$

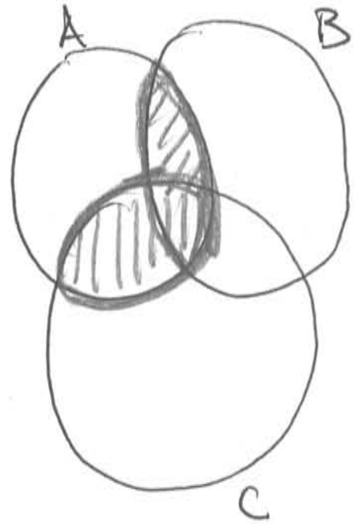
$\therefore \quad F_{i+2} \geq \phi^i$

Pg 80 CLR

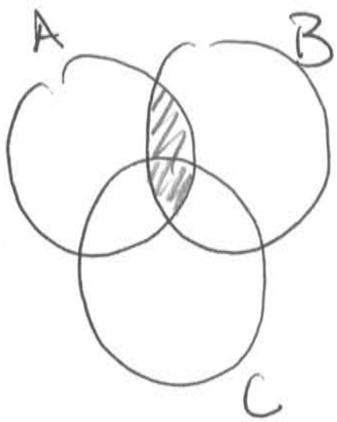
5.1-1 $A\cap(B\cup C) \overset{?}{=} (A\cap B)\cup(A\cap C)$



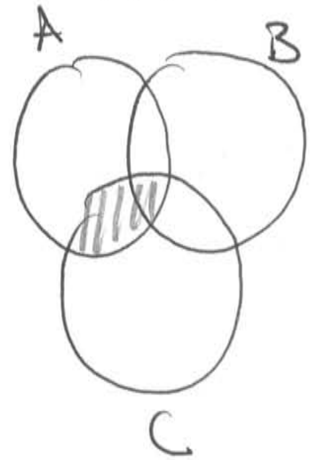
$B\cup C$



$A\cap(B\cup C)$



$A\cap B$



$A\cap C$

5.1-2 $\overline{A\cap A_2\cap\cdots\cap A_n} = \bar{A}_1\cup\left(\overline{A_2\cap\cdots\cap A_n}\right)$

& repeat. on set $\overline{A_2\cap\cdots\cap A_n}$

$\overline{A_1\cup A_2\cup\cdots\cup A_n} = \bar{A}_1\cap\left(\overline{A_2\cup\cdots\cup A_n}\right)$

& repeat on set $\overline{A_2\cup\cdots\cup A_n}$

(4-8)

(a) Given n chips

1	2	output
		Both Good
3	4	
		Both Good

G G ~~Both~~ Good

G B

Chip 1	Chip 2	Chip 1 says	Chip 2 says	
G	G	G	G	(G,G)
G	B	B	G, B	{B,G}, (B,B)
B	G	B, G	B	(B,B) (G,B)
B	B	B, G	B, G	(B,B), (B,G), (G,B), (G,G)

Given n chips one can form $\binom{n}{2}$ pairs to test.

Thus we have for 1 test the following probabilities.

of obtaining

(G,G)	(B,G)	(G,B)	(B,B)	Σ (total)
2 ways	2 ways	2 ways	3 ways	9
2/9	2/9	2/9	3/9 = 1/3	

$2/9 = .\bar{2}$ $\quad 4/9 = .\bar{4}$ $\quad 1/3 = .\bar{3}$

w/ 7 chips

chip #	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							

Lets test two chips: say 1 + 2 if the outcome is (G, B)

$\boxed{B}$ $\boxed{B}$

outcome:

(G, B) then either chip 1 = B + chip 2 = G or chip 1 = B, chip 2 = B

in ~~other~~ either case

But ⇒ that chip 1 is Bad *

If outcome is (B, G) ⇒ chip 1 = G + chip 2 = B or chip 1 = B, chip 2 = B

But ⇒ chip 2 is Bad *

If outcome is (B, B) ⇒ chip 1 = G, chip 2 = B or chip 1 = B, chip 2 = G or chip 1 = B, chip 2 = B

No conclusion

If outcome is (G, G) → chip 1 = G, chip 2 = G or chip 1 = B, chip 2 = B

No conclusion.

$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$ = # of pairs.

For each test that yields a (B,G) or a (G,B) result we can elliminate a chip, from further testing; by knowing that it is bad.

~~...~~

$n = 8$

x x x x x x x x
7 6 5 4 3 2 1

$7\text{ pairs} + \cancel{6} + 5 + \cancel{4} + \cancel{3} + 2 + 1 = 20 + 7 = 27 + 1 = 28$ ✓.

$\frac{8(7)}{2} = 4 \cdot 7 = 28$ ✓

5.1-3

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

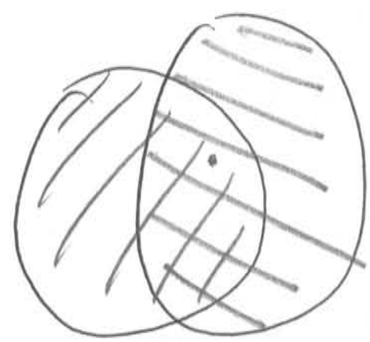
$$- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots$$ (all pairs) $\binom{n}{2}$

$$+ |A_1 \cap A_2 \cap A_3| + \cdots$$ (all triples) $\binom{n}{3}$

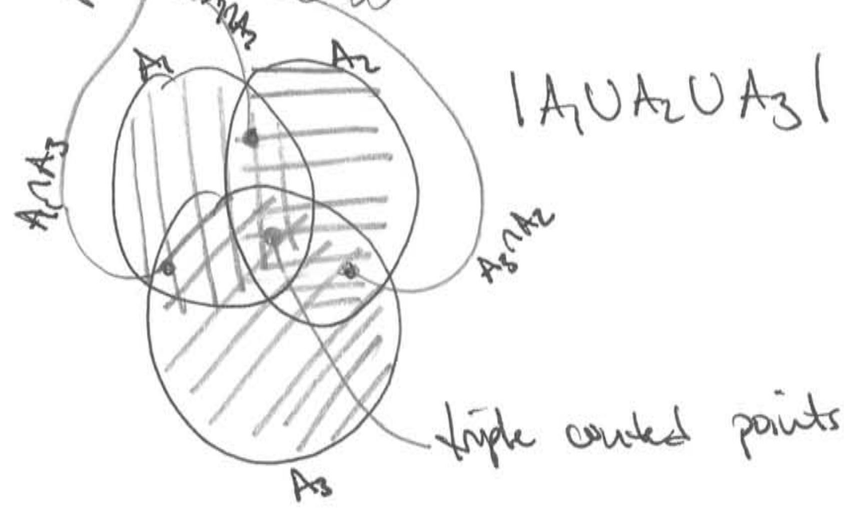
$$- \cdots$$

$$+ (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

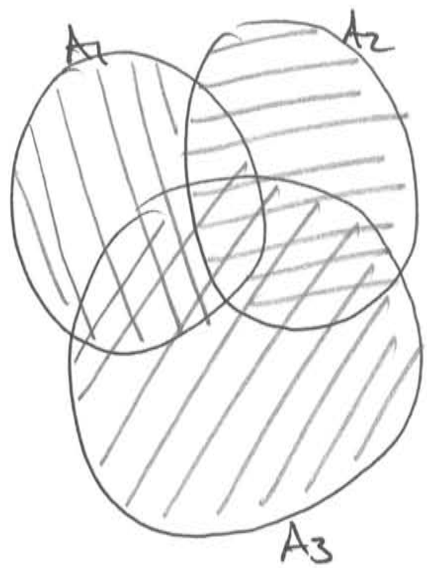
$A_1 \cup A_2$:



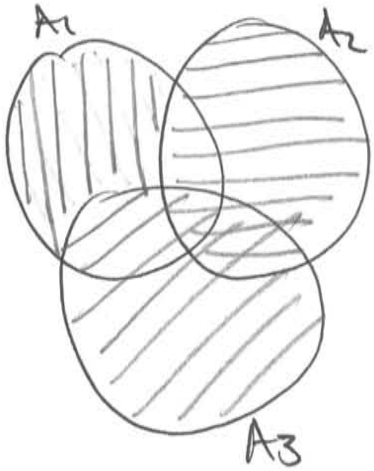
$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



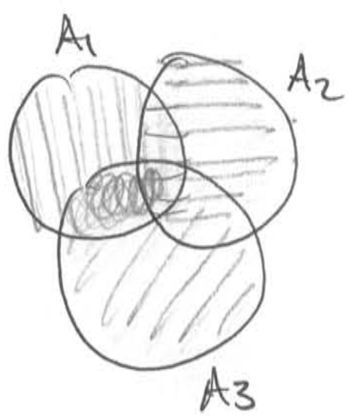
$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$
$$- |A_1 \cap A_2| - |A_1 \cap A_3|$$
$$- |A_2 \cap A_3|$$



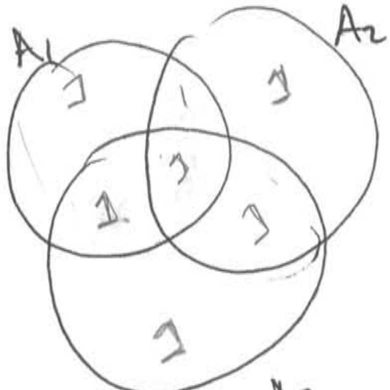
$$|A_1| + |A_2| + |A_3| - |A_1 \cap A_2|$$



$$|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3|$$



$$|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$



$$|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

04-06-023

$\binom{n}{1}$ +

# of sectors.

$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

independent sectors   single intersection sectors   double intersection sectors

$$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$$

idea is some regions are double counted (single intersection $A_1 \cap A_2$)
triple counted (double intersection $A_1 \cap A_2 \cap A_3$)

etc.

the # of elements (cardinality) of the full set would be

$$1 \binom{n}{1} + \frac{1}{2}$$

weight of each element   # of single sets

$\binom{3}{1} = 3$ $\binom{3}{2} = 3$ $\binom{3}{3} = 1$

$\frac{3!}{1!\,2!} = 3$

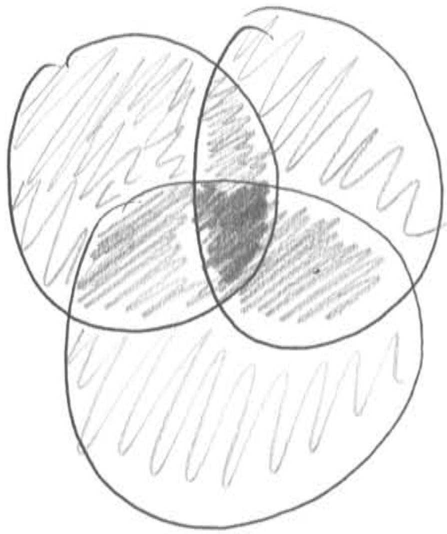
double intersections   triple intersections

$A_1 = A_1^1 \cup A_1^2 \cup A_1^3 = \cancel{A_1^1 \cup (A_1} A_2$ $|A_1| = |A_1'| + 1 \cdot 1 + 1 \cdot 1$

$A_2 = A_2^1 \cup A_2^2 \cup A_2^3$ $|A_2| = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1$

$A_3 = A_3^1 \cup A_3^2 \cup A_3^3$

$|A_1| + |A_2| + |A_3| -$



$|A_1| + |A_2| + |A_3| =$ (every double counted region)

$A_1 \cup A_2 \cup A_3 = A_1 /$

5.1-4 if x is odd $x = 2k+1$ $\forall$ $k \in N$.

$x \in \{1, 3, 5, \cdots\}$

This is a 1-to-1 correspond & .'. the odds are countable

5.1-5 $\{x_1, x_2, \cdots, x_n\}$

# of subsets:

$\phi + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n}$

$\shortparallel$

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n} = (1+1)^n = 2^n$.

5.1-6 By set definition $(a,b) = \{a, \{a,b\}\}$

$\neq$ $(b,a) = \{b, \{b,a\}\} = \{b, \{a,b\}\}$

Thus a set theoretical def of an ordered pair might be

$(a_1, a_2, \cdots, a_n) = \{\cancel{a_1, \{a_2, \cdots, a_n\}}$

$= \{a_1, \{a_1, (a_2, \cdots, a_n)\}\}$

w/ $(a_\phi) = a$

pg 83 CLR

(8.2-1) partial order:

relation that is reflexive, antisymmetric, + transitive

antisymmetric $\Rightarrow$ aRb and $bRa \Rightarrow a=b$

$A \subset A$ so reflexive property holds

$A \subset B$ + $B \subset A \Rightarrow A = B$ so antisymmetric holds

$A \subset B$ + $B \subset C \Rightarrow A \subset C$ so transitive holds

Total order would mean that $\forall$ two elements $A + B \in$ Set

either $A \subset B$ or $B \subset A$.

Take $A = \{1, 2\}$ + $B = \{2, 3\}$ - we see this is not true.

(8.2-2) An equivalence relation is ~~st~~ reflexive, symmetric + transitive

$a \equiv a \pmod n$ yes $a - a = 0 = q \cdot n$ $q = 0$

$a \equiv b \pmod n \Rightarrow a - b = q \cdot n \Rightarrow b - a = -qn \Rightarrow b \equiv a \pmod n$

$a \equiv b \pmod n$ + $b \equiv c \pmod n \Rightarrow a - b = q_1 n$ + $b - c = q_2 \cdot n$

$\therefore$ $a - c = q_1 n + q_2 n = (q_1 + q_2) n = \hat{q} n$

The equivalence classes correspond to ~~β~~ $qn + j$

$\forall j : 0 \le j \le n-1$

04-08-02 ∫

pg 83 CLR

5.2-3

(a) $R = \{(a,b) : a,b \in N$ + $a=b$ or $a=b-1$ or $a=b+1\}$

Then $(a,b) \in R$. aRa ✓ ~~(a,b)~~ ~~aRb~~ aRb $a=b$ Then bRa ✓

or $a=b-1$
$b=a+1$ bRa ✓ so symmetric

$2R3$ + $3R4$ but $4 \not R 5$.

or $a=b+1$
$b=a-1$ bRa ✓

Not transitive

(b) Set inclusion or subsets $A \subset A$ ✓

$A \subset B$ + $B \subset C \Rightarrow A \subset C$ ✓

if $A \subset B \not\Rightarrow B \subset A$.

(c) $\neq$ ~~a~~ $\neq a$ is false

$R = \{(a,b) . a, b \in N$ $a<b$ or $b<a\}$

Then aRa is Not true $\quad aRb$ $bRc \Rightarrow aRc$? No. !!

$a<b$ or $b<a$ $\quad b<c$

5.2-4 R is an equivalence relation on $S \times S$.

so R is reflexive, symmetric, + transitive, if in addition R is antisymmetric.

$[a] = \{x : (a,a) R x\}$. TO show that a is the only element in the set $[a]$

assume there was another element. say b. Then

$(a,a)R(b,b) \Rightarrow (b,b)R(a,a)$ by symmetry $\Rightarrow a=b$ by antisymmetry

$\rightarrow\leftarrow$. only one element $[a]$ or a itself.

04-08-02 1

Pg 83 4R

(8.2-8) No. transition regions give + then

We are only given aRb ~~Not~~ bRc

Wouldn't be correct? ~~But don't see contradiction~~. No definition of transitivity is that aRb + bRc imply aRc. This is not true however? Don't see.

pg 86 CLR

5.3-1

(a) f is injective ⇒ distinct elements of A produce distinct elements i.e. $a \neq a'$ $f(a) \neq f(a')$

Assume $|B| \leq |A|$ + show this leads to a contradiction.

There are $\binom{|A|}{2}$ distinct pairs of points in set A

$$= \frac{|A|(|A|-1)\cdots}{2}$$

1st there must be as many pts in B as in A because if.

$f(a)$ $\forall$ a in A must be distinct ⇒ ∃ f(a) distinct pts

⇒ $|B| \geq |A|$

(b) f is surjective its Range is its codomain.

$\Rightarrow$ $f(A) = B$

since given a + $a' \in A$ $f(a) = f(a')$

we could have fewer elements in B then in A thus

$|B| \leq |A|$

5.3-2 f bijective ⇒ onto + 1-1

$f(x) = x+1$ $N = \{0, 1, 2 \cdots\}$

No since $f(N) \neq N$ since 0 is excluded

thus f is Not onto.

If Domain + codomain are Z yes.

04-11-02 1

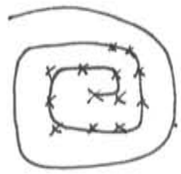
pg 86 CLR

5.3-3

Define R^{-1} $\ni$ if aRb then $bR^{-1}a$

can we show $bR^{-1}a$ is a fn. $\Rightarrow$ ~~given~~ $\forall$ b $\exists$ precisely one a $\ni$ $(b,a) \in R^{-1}$. This is true since f is a bijection

5.3-4 $\mathbb{Z} \rightarrow (\mathbb{Z} \times \mathbb{Z})$



Form a spiral from the origin.

5.4-1

Each person shaking hands can be thought of as forming a graph where each handshake connects two vertices. Since on 1 handshake (one edge) each professor counts 1 or 2 total handshakes have happened ∀ edge

$\sum \text{degree}(v) = 2|E|$ must be true

5.4-2 in an undirected graph self loops are forbidden thus a cycle must start at some vertex go to another vertex & return again giving 3 vertices {start, one away, end} w/ end = start

5.4-3 Simple path means each vertex in the path set is distinct. To obtain a simple path simply don't back track over vertices

$\langle v_1, v_2, ; v_k \cdot, v_p, \cdots, v_n \rangle$

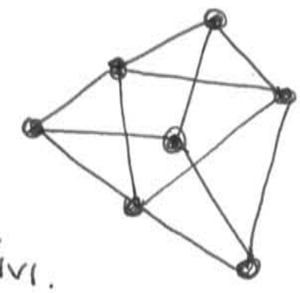
if $v_k = v_p$ then delete the nodes between v_k & v_p to obtain a simple cycle.

Again the same ~~idea~~ idea would work (in both directed + undirected) Graph cases. Simply find the 1st to vertices from the left & from the right that equal & deleting the elements in between

(5.4-4) connected means any two vertices are reachable.

i.e. $\exists$ path between any two vertices.

intuitively there must exist alot of edges.



consider the set of vertices $\{v_1, v_2, \ldots, v_{|V|}\}$

+ the path from $v_1 \xrightarrow{(1)} v_2 \xrightarrow{(1)} v_3 \xrightarrow{(1)} v_4 \xrightarrow{(1)} \cdots \xrightarrow{(1)} v_{|V|}$.

Now in general there will be more then 1 edge between two given vertices. I.E. the path length will be greater than 1.

Thus there has to be at least $|V|-1$ edges, this would have a direct path of 1 edge between any two vertices.

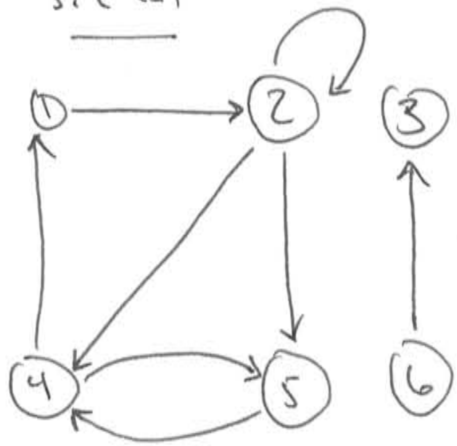
$\therefore \quad |E| \geq |V|-1$

(5.4-5) Given an undirected graph. An equivalence relation on the vertices must satisfy 3 things
1) aRa ✓
2) $aRb \Rightarrow bRa$ ✓ due to undirected
3) aRb and $bRc \Rightarrow aRc$ ✓

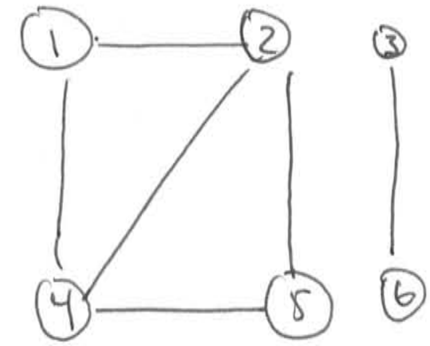
(B) relation #3 will hold for a general directed graph pg 90 CLR 04-15-02

5.4-6

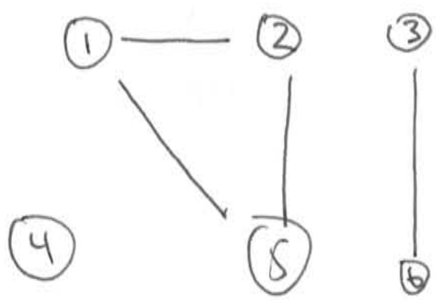
5.2 (a)



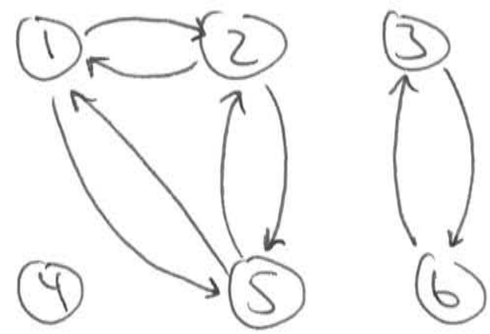
undirected version $\Longrightarrow$



5.2 (b)



directed $\Longrightarrow$ version



? Do I add self loops?

pg 90 CLR

04-15-02 1

5.4-7 hypergraph has each hyper-edge connects an arbitrary subset of vertices

let V_1 = set of vertices in the bipartite graph.

V_2 = hyperedges (set)

Then given ~~$v_1 \in V_1$~~, ~~$v_2 \in V_2$~~ $\cancel{\exists}$ $(v_1, v_2) \in E$.

then